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The Teaching of ARITHMETIC in Primary Schools

K. LOVELL
and C. H. J. SMITH

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The Teaching of Arithmetic in Primary Schools

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PREFACE TO THE FOURTH EDITION

A number of changes have been made in the text, and references to some of the newer developments in the teaching of primary school mathematics have been added.

K. L.

PREFACE

THIS book is primarily addressed to students in training colleges and newly qualified teachers, although it is hoped that it will serve as a handy reference book for teachers generally and for those parents who are interested in the subject.

Its distinctive features include the setting out in detail of the progressive steps in learning a process. It is the conviction of the authors that the key to success in the teaching of mathematics lies in the careful grading of examples, and that the prime reason for including the subject in the school curriculum is to enable children to tackle problems in life which depend upon mathematical knowledge. Reference is also made in the book to recent research in the teaching of arithmetic, to the teaching of elementary geometry, and to backwardness.

We should like to express our thanks to several of our colleagues for helpful suggestions, and particularly to Mr. A. F. Scott, M.A., Lecturer in English at Borough Road College, who read the whole of the typescript and suggested numerous improvements in the text, and to Mr. H. S. Foster, of the University of London Press, for his care and attention to editorial details.

References in the footnotes are frequently used to amplify small points, and the books and journals referred to are not necessarily included in the bibliography.

K. L.
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Borough Road College,
Isleworth, 1953.

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CHAPTER I

EARLY STEPS AND VITAL PROCESSES

The Importance of Pre-school Number Experience

READINESS for any arithmetical or mathematical process seems to depend upon the experience of the child, the nature of the process, the intelligence of the child and his maturity. Parents and teachers of young children can, perhaps, do little about the two last-named factors, but they can do much in respect of the child's experience and the nature of any process presented. It is imperative, therefore, that the parents, the staffs of Nursery Schools, and all who have responsibility for the bringing up of young children should realise the necessity of providing ample opportunities for the child to acquire a readiness for mathematical concepts and relationships. Ideally, these should come from the home and community, and should be the outcome of normal everyday pleasurable experiences in which number, the idea of size, and so forth are incidental.

Many young children on entering school have already gained a considerable knowledge of number from their pre-school life. Buckingham and Machatchy,[1] in a study of over 1,000 children, in the United States of America, entering the first grade at the age of six, found that ninety per cent of the children could count up to ten and about sixty per cent to twenty. These figures may be rather high for children in this country, since here they begin school a year earlier, but they do give an indication that in pre-school years many children acquire the ability to count by rote if not with understanding.

---

[1] Buckingham, B. R., and Machatchy, J.: *The Number Abilities of Children when they enter Grade One*, Twenty-ninth Year-book of the National Society for the Study of Education, 1930, pp. 473–524.

Suggestions for Parents and all Concerned with the Development of the Pre-school Child

The following suggestions illustrate the kinds of activities which should be encouraged in order to help the child to acquire an elementary idea of number, size and shape[1]:

1. Counting the number of children around the table or in the family; counting out the number of knives, forks, and spoons necessary for laying the table; counting the number of buttons on a coat or the number of cows in a field.

2. Reciting nursery rhymes dealing with simple numbers in progressive order, e.g. Ten Little Nigger Boys. The rhymes should be accompanied by the appropriate bodily actions.

3. Arranging objects in order of size, e.g. beads, boxes, chairs. Cutting paper into different lengths for comparison; the handling of shapes and fitting them as insets into a board cut with appropriate holes to receive them.

4. Lifting articles of differing weights to acquire the idea of heavy and light. Use of a toy balance.

First Steps in the Infant School

On entering the Infant School (five to seven plus years), the average child might well have a further six months of work involving toy activities, the handling of sizes and shapes, the chanting of nursery rhymes, and the telling of stories involving number, in order to enrich experience and further consolidate habits which have been learnt in the home and community. For children in whose homes pre-school number experience was limited, there must be adequate and sustained opportunities for acquiring number experience involving a wide range of varied activities. Since most things which come into a child's life can be pressed into the service of counting, there should be little difficulty in selecting suitable exercises. As soon as possible every child should be able to count with understanding. He must also be able to distinguish between the groups of 5 and the position indicated by 5, i.e. the fifth place.

---

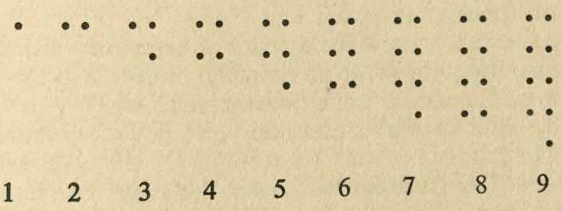
[1] See an important footnote on page 111 regarding Piaget's views on the formation of the concepts of number, quantity, etc.

The Association of a Word with a Group and a Symbol

Soon the symbol which represents a group of objects has to be connected with the articulated sound by which we recognise it. Indeed, three things have to be connected:

1. The figure 3.
2. The spoken and written sound (three).
3. The pattern in which the objects may be arranged.

Begin by arranging groups of bricks or counters and place the appropriate figures, cut from cardboard or metal, underneath. Or, from stiff cardboard cut pieces about 4 inches by 3 inches and on them paint groups of dots with the correct figure underneath. The pattern of dots recommended is:



Such a pattern enables a child to see clearly the relationships between the numbers.

The next step is to teach the child to write the figures as symbols. The methods suggested are:

1. Tracing in a sand-tray or in the air from a picture of a figure provided.
2. Tracing over printed figures, using tracing paper.
3. Completing uncompleted figures. These can easily be prepared by the teacher, and duplicated so that each child has a set of figures on which to work.
4. Using stencils.
5. The Montessori method, in which figures cut out of sandpaper are pasted on to a smooth background such as a smooth block of wood. The child feels around the rough surface of the figure with his fingers. This method employs the visible, tactile, and muscular approaches.

6. The straightforward copying of figures.

Few teachers rely on one method only, but whichever method is being employed at a specific moment, the child must be carefully watched to see that the exercise is being carried out correctly. Close guidance and supervision must also be maintained when the symbols are first written on paper. Without this guidance children tend:

1. To make the figures too small.
2. To reverse figures occasionally, e.g., Ɛ for 3 and ᔦ for 4.

As soon as the child can associate a group with a word and a symbol and can write the symbols, exercises of the following types should be given:

1. The child to count aloud a number of objects and write down the appropriate figure.
2. The teacher to provide a series of cards on which there are pictures (of animals, as an example) or dots, and the child has to write the correct figure underneath.
3. The child to read aloud and write down the date, the number of pages in a book (in reading 231 the child will, of course, say 'Two three one' and not at this stage 'Two hundred and thirty-one'), the numbers on the coathooks in the cloakroom, and the numbers on the registration plates of motor-cars.

Early Addition and Subtraction

Once a child has had ample meaningful experience of number born out of many concrete situations, and can associate a group with a symbol and a spoken word, he is ready for the simplest activities involving addition and subtraction. Addition should be taught first, and should be approached through the use of bricks, counters, dots, or other suitable objects. Two groups, in this case round counters, are laid out as follows:

$$\begin{array}[t]{ll}0 & 0\\ 0 & \end{array} + \begin{array}[t]{ll}0 & 0\end{array} = \begin{array}[t]{ll}0 & 0\\ 0 & 0\\ 0 & \end{array}$$

The signs for 'plus' and 'equals' can be chalked on the table on which the demonstration is taking place or painted on pieces of cardboard; at the same time they must be explained in simple language as 'and' and 'makes' or 'gives' respectively. The child then places the appropriate figure under each group.

It will be noted that at this early stage all addition is set out horizontally, because the layout of material is horizontal at first. Considerable experience of this step will have to be given to each child, and although after some practice he might cease to use objects or pictures of objects, he may for some time continue to use his fingers, or use strokes on a piece of paper, as an aid in the adding of numbers. The teacher should, however, attempt as soon as possible to wean the child away from the method of counting by units; in connection with this, games are most helpful (see end of chapter). The child should be able to write down the answer of 6 + 2 without hesitation. At this point, work-cards may well be introduced. These can be made by the teacher himself, and by this means he can ensure that every child works every possible combination involving addition of numbers which give totals up to and including 9[1]. Much will be said of work-cards in later chapters; it will be sufficient to say here that they are a vital necessity in the Infant School.

In the introductory work in subtraction, where meaning is the chief consideration, it is helpful to use a variety of expressions ensuring a complete understanding of the term. The child's expression for subtraction will be in terms of experiences out of which the need for it grew, e.g. 'I have two more apples than John' or 'I need three more pieces to complete the puzzle'. But as soon as simple subtraction is attempted on paper, it is better at first to keep to a single expression, such as 'take away', to avoid confusing the child.

The actual process of subtraction may be illustrated by using objects, say counters, as were used in connection with the teaching of addition. Lay out groups as indicated below:

---

[1] This should be done as soon as possible. See *Addition and Subtraction Facts and Processes*, Scottish Council for Research in Education, published by University of London Press Ltd, 1948.

$$\begin{array}{ccccc} 0\ 0 & - & 0\ 0 & = & 0\ 0 \\ 0\ 0 & & & & 0\ 0 \\ 0\ 0 & & & & \end{array}$$

and explain the minus sign as meaning 'take away'. The child must then place the correct figures underneath. Proceed as soon as possible to the use of work-cards on which there are exercises with the minuend a maximum of 9.

Once the two processes have been mastered, work-cards should be given which give addition and subtraction exercises on the same card in scrambled order. Finally, the child must be shown how to set down addition and subtraction exercises in vertical form. This is a necessary step preliminary to the teaching of place value. It is thought that there is little to choose between setting down as:

$$\begin{array}{llrcllr} (a) & 4\,+ & \text{or} \quad 4 & \qquad & (b) & 8\,- & \text{or} \quad 8 \\ & \underline{\ 3\ } & \underline{+\,3} & & & \underline{\ 5\ } & \underline{-\,5} \end{array}$$

so long as the plus and minus signs are definitely inserted.

Place Value

By the time the child has reached the stage where he is ready to advance beyond 9, an effort must be made to introduce him to the problem of place value. An approach can be made along these lines. Suggest that a child is to count out twenty-eight matchsticks and that he is to tie them up into bundles of ten as far as possible. A box is provided having compartments labelled *Tens* and *Units*, and he is told that each bundle of ten is to be placed in the *Tens* compartment and the loose sticks in the *Units* compartment. If the box is then suspended against the blackboard and the child records on the board, beneath

0 0	oooooooo
tens	units

the box, the number of bundles of ten and the number of loose matchsticks and the box then taken away, the notation for 28 will remain written as T U.
2 8

It is vital that children have ample experience in using this piece of apparatus in the manner described. There must be adequate experience in bundling in tens and in placing the bundles and odd units into the respective compartments and then writing down the result. To begin with, always select numbers which yield tens and some units, and it is better to refrain from using numbers between ten and twenty because such numbers are named units first then tens, e.g. seventeen. Later, to introduce 20, 30, and so on, it is useful to take a number such as 28 and take away from it one unit at a time. Eventually, the position is reached where there are two bundles of ten but no units, and the complete absence of any figure in the units column is indicated by *nought* or 0.

Similarly hundreds, tens, and units can be made familiar to the child by using a box in which there are three compartments, as shown in the diagram:

O	0000	ooo
hundreds	tens	units

Matchsticks are bundled into tens as before and the odd ones placed in the right-hand compartment. The child is then told to tie together the bundles of 10 into larger bundles each containing ten bundles of 10, any odd bundle of 10 being placed in the centre compartment. The number 143 would then be seen by the child as in the above diagram.

The Use of Games

Games can be of great value as a means of consolidating processes which have been introduced by more formal methods. If children enjoy playing a game and provision is made for frequent participation, much number involved in the game will be learnt incidentally. For although there has been much argument about the value of games, the most common conclusion is that games are of value because they stimulate children into wanting to be able to manipulate number rather than in providing drill itself. Nevertheless, some drill is provided. At this stage, games will provide the experiences out of which will come greater facility to read and write numbers and to operate the simpler of the Primary Addition and Subtraction Facts (see Chapter II).

In games the teacher should note:

1. Avoid games unsuited to the physical development of the children.

2. Avoid games which make use of elaborate mechanical equipment that soon gets out of order; also games which are expensive to buy.

3. Avoid games made of cardboard (except such as Ludo), which will not stand wear and tear. Rather select apparatus and games made of hardboard, wood, or metal, simply yet stoutly constructed, so that they will neither go wrong easily nor wear out without reasonable use.

4. Have enough games of different kinds available so that the whole class can take part in games, at the same time if necessary.

A brief description of a few games is given:

1. *Beanbags.* Beanbags should be made of closely woven material and filled with dried peas or beans. Targets, which can be made of many designs, can be chalked on the floor of the classroom or in the playground. A simple type is illustrated. The game is usually played by allowing each child to throw twice, and his score is the sum of the scores obtained on each throw. To help to fix in the child's mind the result of adding zero to another number, a rule may be made that the score is zero if the beanbag falls on a line and not wholly within a square.

3	5	1
7	2	4
9	8	6

2. *Skittles.* Skittles may be purchased, but old chair rounds painted and numbered are just as useful. Rubber balls or marbles may be used to knock down the skittles.

3. Throwing rubber rings over hooks screwed into boards hung on the wall. Rubbers can be obtained from fruit-jars, and a useful board can be made by driving nails (about 6 inches apart) into a piece of wood and hanging it at about eye-level for the children when playing.

4. A simplified dart-board may be used with suction tips on the darts. On no account should the normal sharp-pointed dart be used.

5. Marbles and balls can be rolled through semicircular arches cut in the lower side of a piece of wood.

6. Miscellaneous games in which dice are thrown to determine how many places to move 'men' or 'counters'; also Dominoes and Tiddlywinks.

7. *Jig-saw Puzzles.* Take two pieces of hardboard of size, say, 9 inches. On one paste a picture, then cut it into 9 squares, and on the back of each piece write a question, e.g. $2+5$? Divide the other piece of hardboard into 9 corresponding squares, and on each of these write in the answers to the questions on the first set of pieces. Before the work begins insist that all the picture pieces are turned face downwards. As each answer is found it is put in its proper place so that when all the answers are in their correct places the picture is formed and is turned over.

8. *Tops.* A hexagon is cut from a square of hardboard of side 6 inches. The six triangles (each of which has the side of the hexagon for base) are brightly coloured and a different number painted on each. A nail or butcher's skewer is passed through the centre of each hexagon. After spinning, the hexagon stops with one side on the table, and the child notes the number on the triangle.

The writer has discussed elsewhere[1] the use of *structural* materials (e.g. those of Dienes, Cuisenaire and Stern) in the teaching of number and number operations.

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[1] Lovell, K.: *The Growth of Basic Mathematical and Scientific Concepts in Children* (University of London Press Ltd, 1961).

CHAPTER II

INFANT SCHOOL—ADDITION AND SUBTRACTION

The 100 Basic Addition Facts

It is hoped that enough has been said in Chapter I to make it clear that much time may be lost in trying to teach the simplest combinations of number involving addition and subtraction unless the children are of a sufficient mental age, and have enough understanding of number obtained through experience, to appreciate what they are doing. Although it is vital that the children should be properly prepared, it is equally important that the teacher should realise how many basic (or primary) addition facts there are and have a plan for teaching them. This is not true of all teachers; thus it is necessary to make it plain at the outset that every possible addition exercise that a child (or adult) is called upon to work depends upon a knowledge of the basic addition facts, and, indeed, progress in arithmetic as a whole is determined by the degree of proficiency attained in knowing the basic facts of addition, subtraction, multiplication, and division. There are one hundred basic addition facts altogether, and they are made up of every number from 0 to 9 inclusive added, in turn, to every number from 0 to 9 inclusive. The facts are of such vital importance that they are listed below:

0 0 0 0 0 0 0 0 0 0
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

1 1 1 1 1 1 1 1 1 1
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

2 2 2 2 2 2 2 2 2 2
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

3 3 3 3 3 3 3 3 3 3
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

4 4 4 4 4 4 4 4 4 4
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

5 5 5 5 5 5 5 5 5 5
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

6 6 6 6 6 6 6 6 6 6
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

7 7 7 7 7 7 7 7 7 7
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

8 8 8 8 8 8 8 8 8 8
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

9 9 9 9 9 9 9 9 9 9
0 1 2 3 4 5 6 7 8 9
– – – – – – – – – –

It will be seen at once that these one hundred facts are made up of forty-five direct facts (e.g. 5 + 8), thirty-six reverse facts (e.g. 8 + 5; obviously facts such as 6 + 6 cannot be reversed) and nineteen facts involving zero, and it cannot be stressed too emphatically that every child must cover every fact, not just once or twice, but until every combination is known. In order to effect this, the following suggestions are made:

1. Use work-cards. Let each set of cards cover ten basic facts and have ten such sets to cover all the facts. Make sure that each child works through each set.

2. Make the children write out (copying from the blackboard), and occasionally recite, composition tables for various numbers, e.g.:

'One and three make four.'
'Three and one make four.'
'Two and two make four.'

3. Ask questions around the class, e.g.:

'Two and two?'
'Three and what are four?'

The teacher must exercise much care and supervision at this stage, and must check the facts known by each child. This is a most worth-while and important task, for many of the errors found later in the Junior School arithmetic would be avoided if the basic facts were known thoroughly in the Infant School. There is of course one danger here which must be guarded against. A child must know ultimately that 'three and two make five', not because the teacher tells him so, or because it is a fact to be learnt by heart, or because it is in the addition table, but because he has discovered the fact through experience. Thus in the case of errors arising, the child must correct them for himself, if necessary by going back to bricks or counters in order to establish the right answer by counting.

Two further points should be noted. First, it is well known that the basic addition facts are not equally difficult to the learner.[1] Wilson[2] has suggested groupings which introduce the harder facts first in order that they may have more drill, but it is doubtful in our present state of knowledge if any worthwhile suggestions can be made regarding the order in which the various facts should be introduced.[1] It can be said, however, that it is unnecessary that all progress should be held up until there is one hundred per cent mastery of the facts. Selected easy addition exercises involving tens and units, but not *carrying* from the units column to the tens column, provide useful practice in covering those combinations which are not well known. Second, some teachers do not recommend the direct teaching of zero facts, e.g.:

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[1] See *Studies in Arithmetic*, Vol. 1 (1939), and Vol. 2 (1941), reports prepared by the Scottish Council for Research in Education (University of London Press Ltd). It is also true that certain basic subtraction, multiplication, and division facts are more difficult than others.

[2] Wilson, G. M.: *100% Drill Books in Fundamental Processes* (Macmillan Co., New York, 1933).

$$\begin{array}{cc} 9 & 0 \\ 0 & 6 \\ - & - \end{array}$$

With this point of view we disagree entirely, and would state emphatically that the zero facts should be committed to memory in the same way as the other facts, for it will be remembered that in each of the one hundred facts real knowledge comes through confirming the fact by counting real things—in the case of a zero fact, six bricks and zero bricks equals six bricks. Thus while it may be true that in one sense combinations involving zero facts should be established by a moment's reflection (six bricks and nothing gives or makes six bricks), it is equally true that for efficient working one should be able to make an automatic response to any combination without consideration or thought.

The 100 Basic Subtraction Facts

There are 100 basic subtraction facts, and it is imperative to realise that all work involving subtraction of number depends upon making an accurate and immediate response to

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[1] Although it is known that the $0+x$ fact is more difficult than the $x+0$ fact and that the $x-x$ fact is more difficult than might have been expected, this does not give positive guidance concerning the order in which the facts should be introduced. See *Addition and Subtraction Facts and Processes* (University of London Press Ltd).

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the basic facts when the latter present themselves. The actual combinations are listed below:

0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0
–	–	–	–	–	–	–	–	–	–
1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1
–	–	–	–	–	–	–	–	–	—
2	3	4	5	6	7	8	9	10	11
2	2	2	2	2	2	2	2	2	2
–	–	–	–	–	–	–	–	–	–
3	4	5	6	7	8	9	10	11	12
3	3	3	3	3	3	3	3	3	3
–	–	–	–	–	–	–	—	—	—
4	5	6	7	8	9	10	11	12	13
4	4	4	4	4	4	4	4	4	4
–	–	–	–	–	–	—	—	—	—
5	6	7	8	9	10	11	12	13	14
5	5	5	5	5	5	5	5	5	5
–	–	–	–	–	—	—	—	—	—
6	7	8	9	10	11	12	13	14	15
6	6	6	6	6	6	6	6	6	6
–	–	–	–	—	—	—	—	—	—
7	8	9	10	11	12	13	14	15	16
7	7	7	7	7	7	7	7	7	7
–	–	–	—	—	—	—	—	—	—

8	9	10	11	12	13	14	15	16	17
8	8	8	8	8	8	8	8	8	8
–	–	–	–	–	–	–	–	–	–

9	10	11	12	13	14	15	16	17	18
9	9	9	9	9	9	9	9	9	9
–	–	–	–	–	–	–	–	–	–

Attempts have been made (chiefly in the U.S.A.) to find out which subtraction facts are inherently more difficult. To date no results have been obtained that would enable us to offer suggestions concerning the order in which the facts should be taught. The important points to note are:

1. All the facts should be covered using work-cards.
2. There should be writing out and occasional recitation of the subtraction table.
3. There should be plenty of oral questioning such as, 'Seven take away four equals?'; also the reverse, 'Four and three make?'
4. The zero facts should be included.

The Addition and Subtraction of Larger Numbers

If it is decided to introduce the children to exercises involving the addition of a single column of digits, it is strongly advocated that such columns should be kept short at first to not more than three digits, e.g.:

6
5
8
—

Larger columns lead to errors in the early days of addition, firstly because the child can easily lose the place (especially if similar figures are involved); and secondly because he has to add pairs of numbers, only one of which is seen, as in the example above, where the child has either to add thirteen and six or eleven and eight. While this type of exercise which involves a limited number of digits may be useful, far more profitable are examples which involve the adding of tens and units (without *carrying*), e.g.:

$$\begin{array}{rcc} & T & U \\ & 1 & 4 \\ + & 2 & 3 \\ \hline \end{array}$$

Here it is useful to go back and revise the earlier work on the tens place and to allow the children to count out fourteen objects and twenty-three objects and total; also to bundle them into groups of tens and units. Such revision welds into a meaningful whole *thirty-seven* obtained by counting, *three tens and four units* obtained by counting and *three tens and four units* obtained on paper. Adequate experience (the amount will depend on the child) must then be given of such examples. There is nothing to choose between the method of starting with the bottom figure and working upwards or vice versa, but in all arithmetic a check should be applied whenever possible so that the child can test his accuracy. So in these examples the habit of checking the working by starting from the top and working down should be inculcated (working up, of course, if the original working was made from top to bottom). Finally, exercises involving three digit numbers may be introduced as soon as individuals are ready for them, e.g.:

$$\begin{array}{rccc} & H & T & U \\ & 1 & 3 & 6 \\ + & 2 & 5 & 0 \\ \hline \end{array}$$

In the case of subtraction, exercises must be set which do not involve *adjusting*,[1] e.g.:

$$\begin{array}{rr} \text{T} & \text{U} \\ 3 & 8 \\ -\,1 & 7 \\ \hline \end{array}$$

Again, apparatus may be used so that the children can actually perform the *taking away*, and so verify the results they obtain on paper. As soon as the children are ready, progress to examples which contain hundreds, tens, and units.

Separated or Mixed Addition and Subtraction

There is a tendency in present practice to mix addition and subtraction exercises even at an early stage, but it is suggested that the evidence on which this is based is very weak. Repp[2] claims that mixed drill is more efficient than isolated drill, but his evidence is inconclusive. What he does show is that frequent drill is more effective than infrequent drill. Since, however, drill is usually connected with early learning, it looks as if the child is helped if addition and subtraction exercises are kept apart at first. When each type has been well practised, then introduce examples where the addition and subtraction exercises are mixed.

Extent of Work in the Infant School

It will be noted that in this book examples involving *carrying* in addition and *adjusting* in subtraction are not introduced until Chapter V; that is, in the first chapter devoted to work in the Junior School. It should not be inferred from this that such work should be deferred until the age of seven, but rather that for children of average or below average ability such work is probably better postponed until the first year of the Junior School. Obviously the more able children should be introduced to processes as soon as they can do them with a fair chance of success.

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[1] We are not in favour of using the term *adjusting* in actual teaching, but here and elsewhere it is used in this book for the sake of brevity.

[2] Repp, A. C.: *Mixed versus Isolated Drill Organisation*, Twenty-ninth Year-book of the National Society for the Study of Education, 1930, pp. 535–50.

CHAPTER III

MULTIPLICATION AND DIVISION

FEW surveys have been made in the British Isles connected with the problem of which of the four rules applied to number occurs most frequently in everyday use. The many surveys that have been made in the United States, however, all agree in giving first place to multiplication.[1] Teachers should therefore see that the multiplication process is thoroughly mastered.

Multiplication to be Introduced through Continued Addition

Multiplication is best regarded for the beginner as a special case of addition in which all the numbers to be added are the same. Furthermore, as in addition the use of concrete material preceded the learning of the basic facts, so in multiplication a variety of experiences should come before the learning of the multiplication tables. It may be stated, then, at the outset:

1. Introduce multiplication through continued addition.
2. Let the first experience of multiplication come through practical activities, using concrete material.

A typical activity to illustrate the second point is as follows. Working with a group of three or four children, allow each to draw counters from a pool and place them in piles on the desk. Have a number of cards available which illustrate the multiplication facts, e.g. $\boxed{3 \times 2} = \boxed{6}$, and here let each child place in front of him the card illustrating the fact 1 × 2. Each child then finds and places after the *equals* sign another card having the figure 2 on it (i.e. $\boxed{2}$). Thus each child has in front of him one pile of two counters and two cards which illustrate the fact $1 \times 2 = 2$:

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[1] Wilson, G. M.: *What Arithmetic Shall We Teach?* (Houghton Mifflin Co., Boston, 1926). Page 14 may be cited as a typical reference.

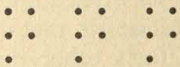
$$\boxed{1 \times 2} = \boxed{2}$$

The children read the cards as *one two equals two*. Each child then takes a further two counters and makes two piles each of two. By counting he finds that he has four counters altogether, and next he finds the appropriate cards to place alongside the piles, giving $\boxed{2 \times 2} = \boxed{4}$, after, of course, it has been emphasised that there are now two groups (piles) each of two counters in front of each child. Let the activity continue until each child has five groups (piles) each of two counters in front of him with the appropriate cards alongside at each stage. Further activities can be worked out for other combinations of numbers. The essential point for the teacher to grasp is that at this stage the children are not being asked to remember that $3 \times 2 = 6$, etc., but that 3×2 means putting out 2 counters 3 times and that by using their knowledge of counting in groups (here 2, 4, 6) they can get the answer. Sufficient experience must be given using concrete material, pictures, and rhymes to establish the meaning of multiplication so that the child reads the card 4×2 as *four twos* or *four multiplied by two*, knowing in fact that he is computing the value of four groups of two. Work-cards should then be introduced on which there are examples yielding products of up to about twelve. A typical work-card would be:

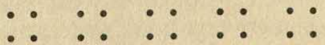
$1 \times 4 =$
$2 \times 3 =$
$5 \times 2 =$
$6 \times 1 =$
$4 \times 3 =$
$2 \times 1 =$

and the answers found by the aid of apparatus.

When the steps enumerated above have been consolidated, the work should be extended to include products of up to about 30. To this end the crate of school milk bottles provides an ideal medium for counting in groups of 5; similarly, arms can be used for counting in groups of 2 and arms and legs for counting in groups of 4. The children can also build up patterns of dots and put in the appropriate multiplication underneath:



$3 \times 5 = 15$



$5 \times 4 = 20$

This must be followed in turn by the use of work-cards which give the factors of numbers up to about 30. To help the child to memorise the facts of group counting (i.e. the multiplication facts) games should be used, as these encourage both speed and accuracy. As an example, *Number Snap* is most helpful, for a child soon learns that he must know his facts and concentrate if he is to call *snap* when he turns over the card 3 × 4 and his opponent turns over 6 × 2.

Should Multiplication Tables as such be Taught in the Infant School?

The methods recommended for the actual teaching of the multiplication tables will be set out in Chapter VI, but a brief reference will be made here concerning the question of whether or not the multiplication tables should be taught in the Infant School. The more able children in the Infant School should be encouraged to learn their tables by more formal methods, for it is as wrong and as stupid to frustrate, and waste the time of, able children by holding them back as it is to frustrate the less able ones by attempting to push them beyond what their mental age naturally allows. No rigid rule, then, can be laid down. What can be said with certainty is that much experience in working multiplication exercises is necessary before calculation becomes accurate and automatic, and unless this is so all later work in arithmetic will suffer. Whenever a child has an understanding of multiplication through continued addition (using apparatus), and has acquired some facility in group counting, there is no reason whatever why the child should not arrange his facts in numerical order; that is, in table form. Indeed, some children learn the facts best through this method, for the building up and writing out of a multiplication table is one of the best ways of learning it. Finally, there is a point of view, which if not openly expressed, is yet often implied by some modern text-book writers and educationists, that there is a way of acquiring the multiplication facts other than by learning them. Nothing could be farther from the truth. Mastery of the multiplication facts calls for hard work on the part of the teacher and of the child.

Division to be Introduced through Sharing

Division is best introduced through the idea of sharing, because this idea is the closest we can get to abstract dividing. Divide the class up into groups of, say, three and give out six counters to each group. Appoint a leader in each group (it is best to appoint the most intelligent children to these posts at first) and let each leader share the counters among the three of them (this includes the leader himself), so that each child gets two counters. Vary the number of children within the groups and the number of counters given out, also make sure that each child gets the opportunity of doing the sharing in turn. After enough experience has been given of these and other similar activities, explain to the children that written exercises can be made about these sharing games. With the aid of the blackboard or prepared diagram, show that the process of sharing six counters among three children may be written on paper as $6 \div 3 = 2$, the sign $\div$ standing for or meaning *shared by* or *divided by*. Later, perhaps in the next period devoted to number work, the class should be divided up into groups as before, and the sharing process using apparatus gone through again. But on this occasion, when each set of counters has been shared, the children write in their books the symbols expressing the process they have been engaged upon.

The next step is the use of work-cards employing dividends of not more than about 12, e.g.:

$8 \div 4 =$
$3 \div 1 =$
$10 \div 2 =$
$6 \div 2 =$
$9 \div 3 =$
$11 \div 1 =$

The examples are worked with apparatus at first; for instance, when working the first exercise on the card the child divides eight counters into four piles each of two counters.

Arising out of adequate experience along the lines indicated (and presupposing a knowledge of the simpler multiplication facts), there will gradually emerge the realisation that the answer to a division exercise can be found by finding the number of smaller quantities contained within the larger. The child will thus come to know that $12 \div 4$ means *How many groups of three in twelve?* If he knows his multiplication facts, the answer will be readily available; if not, he must count out groups of three dots, or he must share out twelve counters among four people and find *practically* that $12 \div 4$ yields four groups of three counters.

CHAPTER IV

THE SHOP, SIMPLE FRACTIONS, AND TELLING THE TIME

Familiarisation with Money

WHEN the child of average ability reaches the age of 6 + years he should be introduced to money, and it is strongly recommended that cardboard money be purchased for this purpose. To those who would advocate the use of real money we would reply that to have available for each child in a class of forty the modest sum of one shilling, one sixpence, one threepenny-bit, six pennies, and six halfpennies, a sum of £5 is required; an amount of money too great for an infant teacher to have lying around the classroom.

The First Step

The first step is to make the child familiar with the various coins. Most children at this age will have a little understanding of the penny, and it is most useful to deal with the other coins in terms of this one. For example, make clear that two halfpennies will buy the same number of sweets as one penny, and that six pennies will buy the same as a sixpenny piece. The children should exchange with one another and with the teacher two sixpences for a shilling, three pennies for a threepenny-bit, two shillings and a sixpence for half-a-crown and so on. With the more able children notes may be used and exchanged in terms of shillings, but with the others it is better to limit the amount to half-a-crown to begin with. It is well worth while to explain why it is that we talk of pence and write *d*, to give the origin of the word shilling, and to tell the children why we talk of pounds and write £. Later in the Junior School, the reasons for paper money and the milled edge of silver coins should be discussed.

The Shop

The classroom shop in the Infant School serves two purposes:

1. It is a device to enable children to acquire further knowledge and experience of money values.

2. It provides scope for imaginative play.

The wise teacher will not attempt to restrict the use of the shop to the first-mentioned use, for knowledge of money values so acquired will surely be incorporated into play.

At this stage the shop should be very simple in construction. A table set apart in the corner of the room with perhaps a pair of home-made scales are the only two large pieces of apparatus necessary. If the shop is to be a grocer shop, get the pupils to bring empty cartons (e.g. corn-flake cartons, sugar cartons, etc.); if a sweet shop, use beads, dried peas, and beans; if a greengrocer shop, some apples, pears, bananas, and so forth, made from Plasticine and coloured. The most important thing is the way the teacher is to use the shop, and as far as directed activities are concerned (i.e. use other than for play), the following suggestions are made:

1. Apart from very occasional class demonstrations, let a group of not more than about five children use the shop at one time, and let one group per day keep to the shop during the number period. See that the other children are otherwise engaged.

2. Let each child in turn be the shopkeeper.

3. Make the children construct price tickets to be attached to the goods. This might be done in, say, a handwork or other period, e.g. $\boxed{4^{d}}$. Some teachers like to place the appropriate number of dots on the card beside the figure, e.g. $\boxed{4^{d}\ ::}$, but this should not be necessary if adequate experience has been given of associating a symbol with a group (see Chapter I). Restrict the price of any one article to sixpence to begin with and have whole pennies only.

4. Each child then in turn buys an article and hands over the appropriate money to the shopkeeper.

5. Each child purchases two articles (so that they together cost less than one shilling) and counts out the total number of pennies required.

6. The pupils transfer the cost of the articles which have been purchased to paper and do the addition; e.g.:

$$\begin{array}{r} \mathit{d.} \\ 3 \\ +\,5 \\ \hline 8 \\ \hline \end{array}$$

7. When there has been ample practice of simple shopping, together with the transference of the transactions to paper, then consolidate the work in addition of money by the use of work-cards but without the use of the shop.

Subtraction may now be introduced through the idea of giving change. For example, if a certain article costs 4*d.* and the child tenders 6*d.* to the shopkeeper, what change must be given? After an understanding of the process has been obtained, the transaction should be transferred to paper, and this in turn followed by work-cards.

It is now a simple step to the addition and subtraction of shillings and pence without carrying or borrowing. Arrange for the prices of the articles to be such that when the purchase is restricted to two items the total of the pence column does not exceed elevenpence. The transactions are later placed on paper, and this followed by work-cards on which are examples of the type:

$$\begin{array}{rr} \mathit{s.} & \mathit{d.} \\ 1 & 3 \\ +\,2 & 8 \\ \hline \end{array} \qquad\qquad \begin{array}{rr} \mathit{s.} & \mathit{d.} \\ 8 & 9 \\ -\,4 & 6 \\ \hline \end{array}$$

With the more able children the teacher can introduce halfpennies, also carrying and adjusting between halfpence and pence and the pence and shillings columns. Should this be the case, full details of the process steps are given in Chapter VIII.

Simple Fractions in the Infant School

Children should be introduced to the objective meaning of the terms *one-half* and *one-quarter* while they are in the Infant School. At first these are terms which they will hear outside school, but more important is the fact that the earlier the objective treatment of fractions begins the better the chance of children getting to understand them, and of achieving success later when they have to manipulate fractions on paper. What, then, should be done in the Infant School? Nothing more and nothing less than the provision of adequate opportunities and experiences to enable the child to understand objectively the meaning of one-half and one-quarter. This can be done by calling upon half the class to stand up, giving out books to half the class, giving out equal-sized pieces of newspaper with the instruction, 'Divide your paper into halves. Crease your paper then and show quarters'; the putting together of halves and quarters to make wholes and so on. Make use of fractions, incidentally, whenever possible.

Telling the Time

The ability to tell the time is a most useful adjunct even for a seven-year-old, and provision should be made in the Infant School for suitable instruction to be given. The following procedure has been found very effective. Have two large clock faces available (at least 2 feet in diameter), with hands which can be moved, made out of hardboard. On the one face have the hours marked only, while on the other have both the hours and the minutes. The words *Past* and *To* should be written in the appropriate places on the face. The teaching steps are then as follows:

1. Hours only. The hands are manipulated to give ample experience in reading the time in hours only as indicated by the

little hand. It will naturally be pointed out that at the hour the large hand is always at 12.

2. Half-hours. The little hand midway between two consecutive hours and the large hand at 6 or half-way round the face.

3. Quarter past and quarter to the hour. The work done in fractions will help the child at this point.

4. Reading off the time in minutes past or minutes to the hour.

Thus, to begin with the children start with the face which has hours only marked on it, and as soon as they become ready for it the other clock face is brought in. If there is a real clock in the classroom or school hall, or a wrist-watch available, refer to that at each step. Written examples (clock *exercises*) of two kinds can be given:

1. The teacher reads out or writes on the blackboard various times, and the children have to draw a clock face in their books and place the hands in the correct position.

2. The teacher manipulates the hands of the clock face and the pupils write the time in their books, e.g. 12 minutes past 2.

Although it is impossible to get young children to grasp the idea of one hour (the unit of time is too large), something can be done to get across the idea of one minute of time. The children can look at, of course, the seconds hand of a watch or clock, but a very simple experiment for all to see is as follows. Attach a piece of string to a nail in the wall, and at the other end of the string tie a small piece of metal to act as a *bob*, arranging matters so that the bob can swing freely and the distance from the nail to the bob is 39 inches. Let the children count sixty swings or thirty complete oscillations, and point out that that period of time is one minute.

Although stress has been laid on the importance of teaching children to tell the time, it must not be assumed that because they can do this they necessarily have a concept of time in which instants and intervals are co-ordinated in the mind. (See Lovell, *op. cit.*)

CHAPTER V

JUNIOR SCHOOL—ADDITION AND SUBTRACTION

ENOUGH has been said in Chapter II about addition of number which does not necessitate carrying from the units to the tens column. Everything goes smoothly for the child until an example such as:

$$\begin{array}{rcc} & \text{T} & \text{U} \\ & 2 & 7 \\ + & 3 & 5 \\ \cline{2-3} \end{array}$$

is met for the first time. Once again it is profitable, especially with the less able children, to go back to the use of counters or matchsticks and let the child actually count up and find the total of 62. After discussion, the teacher can then work the sum on the blackboard as:

$$\begin{array}{rcc} & \text{T} & \text{U} \\ & 2 & 7 \\ + & 3 & 5 \\ \cline{2-3} & 5 & 12 \\ \hline\hline \end{array}$$

which, of course, is perfectly correct, but it should be pointed out that although strictly correct it is natural to combine all the tens together, and so actually we obtain:

$$\begin{array}{rcc} & \text{T} & \text{U} \\ & 2 & 7 \\ + & 3 & 5 \\ \cline{2-3} & 6 & 2 \\ \hline\hline & & 12 \end{array}$$

as the child has actually found by counting up the bundles of ten. By this means introduce the idea of *carrying*, that is to say, combining all the tens together. Indeed, at first it is a good idea to allow the child to write the sum of the units column below the units column, then transfer the units part of the sum to the units place in the total, and finally add the tens to the other tens in the tens column. As soon as possible, however, the child must be made to sum the units column without writing the total down, think of the notation for it, and carry forward the number of tens to the next column. Some children prefer to write in the carrying figure at the foot or at the top of the tens column. This should not only be permitted but encouraged at first. Experience shows that it is a positive aid in working the examples correctly.

Listed now are some of the process steps which should be covered by every child:

$$\begin{array}{rr} \mathrm{T} & \mathrm{U} \\ 2 & 4 \\ +3 & 7 \\ \hline \end{array}$$

Units and tens columns each of two figures.

$$\begin{array}{rr} \mathrm{T} & \mathrm{U} \\ 3 & 7 \\ +4 & 3 \\ \hline \end{array}$$

Units and tens columns each of two figures, units column adding to 10.

$$\begin{array}{rr} \mathrm{T} & \mathrm{U} \\ 3 & 5 \\ 1 & 7 \\ 2 & 2 \\ +1 & 7 \\ \hline \end{array}$$

Units and tens columns each of three or more figures with units column summing to more than 20.

$$\begin{array}{rr} \mathrm{T} & \mathrm{U} \\ 1 & 6 \\ 2 & 5 \\ +3 & 9 \\ \hline \end{array}$$

Units and tens columns each of three or more figures with the units column summing to 20 or 30, i.e. a multiple of 10, so that 0 has to be inserted in the units total.

$$\begin{array}{rcc} & \mathrm{T} & \mathrm{U} \\ & 1 & 4 \\ & & 8 \\ + & 1 & 7 \\ \hline \end{array} \qquad \begin{array}{rcc} & \mathrm{T} & \mathrm{U} \\ & 2 & 4 \\ & 3 & 0 \\ + & 1 & 7 \\ \hline \end{array}$$

Units and tens columns each of three or more figures with gaps in the tens column or zeros in the units column.

Lastly the work must be extended to numbers involving hundreds, tens, and units, and finally to thousands.

Typical Errors in Addition

An examination of examples worked by children shows that the following errors are fairly common:

1. The basic addition facts are not thoroughly known, e.g. $9 + 7 = 15$.
2. An error in connection with the carrying figure. The latter may be omitted, added in twice, or the wrong number carried.
3. Numbers in the column may be actually omitted when adding, or a number may be added in from another column.

Subtraction

The child should at this stage be in a position to work examples of the type:

$$\begin{array}{rcc} & \mathrm{T} & \mathrm{U} \\ & 4 & 7 \\ - & 2 & 3 \\ \hline \end{array}$$

which does not involve a units figure in the subtrahend larger than that of the minuend. The difficulty arises when this is not so, and an analysis of the methods used to overcome this difficulty shows that, in the main, they are variations of two distinct techniques:

1. Subtraction by Decomposition.
2. Subtraction by Equal Additions.

Method of Decomposition

Begin by placing an example such as—

$$\begin{array}{rr} \text{T} & \text{U} \\ 4 & 2 \\ -2 & 5 \\ \hline \end{array}$$

on the blackboard, and bring the attention of the children to the difficulty before them. Let them bundle up matchsticks so as to make four bundles of ten and two units over. Explain that the bundles can be recounted so as to give three bundles of ten and twelve units over, and let the children actually perform the operation so that with the new arrangement of the matchsticks they can easily subtract two bundles of ten and five units, giving one bundle of ten and seven as a remainder. Finally demonstrate that the exercise is written on paper as:

$$\begin{array}{rr} \text{T} & \text{U} \\ {}^{3}\cancel{4} & {}^{1}2 \\ -2 & 5 \end{array}$$

At first the writing in of the crutches should be insisted upon in order to stamp the method upon the mind of the child. When, however, the child is ready to drop the crutches, he simply says to himself:

'Five from twelve seven, two from three one.'

Later bring in examples involving hundreds, tens, and units. The working becomes:

$$\begin{array}{rrr} \text{H} & \text{T} & \text{U} \\ {}^{3}\cancel{4} & {}^{1}\cancel{2} & {}^{1}5 \\ -1 & 7 & 8 \\ \hline \end{array}$$

Method of Equal Additions

This method of subtraction is best introduced by the use of simple examples, which illustrate the truth that the difference between two numbers is unaltered if the same number is added to each. For example, if Mary is nine years old and Tom is seven, how old will each be in ten years' time and what will the difference in their ages be then? Then apply the principle to an example such as:

$$\begin{array}{rr} \text{T} & \text{U} \\ 4 & 2 \\ -2 & 5 \\ \hline \end{array}$$

Explain that we cannot take 5 from 2, but that we can add any number we like to the minuend providing that we add the same number to the subtrahend and the difference will be the same. If we add 10 (10 covers all possible cases) to the top line in the form of 10 units and 10 to the bottom line in the form of one ten, this will suit our purpose admirably.

Thus the example becomes:

$$\begin{array}{rr} \text{T} & \text{U} \\ 4 & {}^{1}2 \\ -{}^{3}\cancel{2} & 5 \end{array}$$

Once again crutches should be used until such time as the process has become quite mechanical and the child says to himself:

'Five from twelve seven, three from four one.'

In the case of hundreds, tens, and units the example will be written as:

$$\begin{array}{rrr} \text{H} & \text{T} & \text{U} \\ 4 & {}^{1}6 & {}^{1}3 \\ -{}^{3}\cancel{2} & {}^{9}\cancel{8} & 9 \\ \hline \end{array}$$

The Method of Subtraction to be Taught

Considerable research has been made to decide which of these methods is the better one to use. Ballard[1] (England) maintains that with boys and girls the method of equal additions is quicker and fewer errors are made, although the advantage becomes less as the child becomes older (by about thirteen years of age). Other workers have also claimed it to be the better method. However, the well-designed experiment of Brownell and Moser[1] in the U.S.A. suggests that while the method of equal additions might be better for those whose arithmetic concepts are well developed, it is not the better method for other children. The method recommended, on balance, to be tried first is that of equal additions, for with younger children the mass of opinion is that although it is less easy to demonstrate in concrete form it: (*a*) produces fewer errors; (*b*) takes less time; (*c*) is less awkward to work when the minuend has one or more zeros in it.

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[1] Ballard, P. B.: 'Norms of Performance in the Fundamental Processes in Arithmetic with Suggestions for their Improvement', *Journal of Experimental Pedagogy*, 2: 386–405, 1914; 2: 9–20, 1915.

Process Steps in the Teaching of Subtraction

There are a number of pitfalls in subtraction, and teachers must arrange their examples so that as many as possible of the 'snags' are covered. Listed below are a number of process steps which should be covered, although the list is not to be taken as exhaustive:

1. Simple subtraction, no adjusting, but gap in subtrahend; zero in subtrahend; zero in last figure not brought down; zero in answer, e.g.:

$$\begin{array}{ccc} & \mathrm{T} & \mathrm{U} \\ & 3 & 8 \\ - & & 7 \\ \hline \end{array} \qquad \begin{array}{cccc} & \mathrm{H} & \mathrm{T} & \mathrm{U} \\ & 4 & 2 & 9 \\ - & 3 & 0 & 7 \\ \hline \end{array} \qquad \begin{array}{cccc} & \mathrm{H} & \mathrm{T} & \mathrm{U} \\ & 6 & 9 & 7 \\ - & 6 & 4 & 3 \\ \hline \end{array} \qquad \begin{array}{cccc} & \mathrm{H} & \mathrm{T} & \mathrm{U} \\ & 5 & 6 & 8 \\ - & 3 & 6 & 2 \\ \hline \end{array}$$

2. Adjusting in units or tens only with gap in the subtrahend and with one or more zeros in the minuend, e.g.:

$$\begin{array}{cccc} & \mathrm{H} & \mathrm{T} & \mathrm{U} \\ & 4 & 8 & 2 \\ - & & 5 & 7 \\ \hline \end{array} \qquad \begin{array}{cccc} & \mathrm{H} & \mathrm{T} & \mathrm{U} \\ & 4 & 5 & 0 \\ - & 2 & 1 & 8 \\ \hline \end{array} \qquad \begin{array}{cccc} & \mathrm{H} & \mathrm{T} & \mathrm{U} \\ & 6 & 0 & 3 \\ - & 4 & 2 & 1 \\ \hline \end{array}$$

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[1] Brownell, W. A., & Moser, H. E.: *Meaningful v. Mechanical Learning. A Study in Grade III Subtraction.* Duke University Press, 1949.

3. Adjusting in both the units and tens columns, including the case where the middle figure in the subtrahend is 9, e.g.:

$$\begin{array}{rccc} & H & T & U \\ & 4 & 2 & 3 \\ - & 2 & 8 & 5 \\ \hline \end{array} \qquad \begin{array}{rccc} & H & T & U \\ & 5 & 0 & 0 \\ - & 2 & 5 & 7 \\ \hline \end{array} \qquad \begin{array}{rccc} & H & T & U \\ & 7 & 0 & 3 \\ - & 2 & 9 & 4 \\ \hline \end{array}$$

Typical Errors in Subtraction

1. Incomplete knowledge of the basic facts, e.g. $17 - 9 = 9$. Zeros in the minuend or subtrahend seem to be especially likely to cause error, e.g.:

$$\begin{array}{rc} & 6 \\ - & 0 \\ \hline \end{array} \qquad \begin{array}{rccc} & 1 & 0 & 7 \\ - & & 2 & 4 \\ \hline \end{array}$$

as do the subtraction of like numbers in minuend and subtrahend, e.g.:

$$\begin{array}{rc} & 6 \\ - & 6 \\ \hline \end{array}$$

2. Adjusting where adjusting is unnecessary and failing to adjust when needed.
3. Adding instead of subtracting or subtracting figure of minuend from subtrahend.
4. Adjusting 2 or 3 instead of 1, or adjusting from the second figure to the left instead of the first to the left.

The Use of Crutches

Throughout this book the use of crutches has been advocated whenever they will help the young pupil to grasp the essential process and render it mechanical. Many would disagree with this point of view, but here, as in so many other problems in education, there is much opinion but little research to help make a decision. Brownell[1] says crutches are criticised because:

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[1] Brownell, W. A.: 'Place of Crutches in Instruction', *Elementary School Journal*, 34: 607–19, 1924.

1. They confuse rather than clarify.
2. They are sometimes as difficult to learn as the processes they are supposed to simplify.
3. They outlive their usefulness.
4. They encourage the learner to stay at a low level of intellectual activity.

Brownell tries to counter these arguments, but on the whole he is not very convincing. It may be said that before positive suggestions can be offered, the teacher must be guided by the ability of the children. Able children will easily drop the crutches when they feel that they need them no longer; children of average ability find the crutches helpful and take longer to discard them; the least able may never be able to operate successfully without them. If the teacher would eliminate the crutches when they are no longer needed, there could be little complaint of their outliving their usefulness. If the child cannot dispense with the crutch it is better to keep it, for the aim of the teacher must ever be to help the children to get their answers right.

CHAPTER VI

MULTIPLICATION OF NUMBER

IN all branches of arithmetic, but particularly in multiplication, the key to successful work is table accuracy. A good deal of hard work will be necessary, and there is no escape from this in the lower part of the Junior School. Fortunately young children do not look upon the task in quite the same way as their teacher; they love repetition and they have plenty of energy available. The old-time method of learning tables was to chant them orally, varying the order of the elements in the table from time to time to ensure that the whole of the earlier part of the table did not have to be said to arrive at a particular point. The reaction to this rather stereotyped method was to abolish the learning of tables altogether and to provide the pupils with ready reckoners. This was surely one of the greatest mistakes ever made in the teaching of arithmetic; children certainly do not pick up a knowledge of arithmetical tables incidentally.

Chanting in chorus, particularly in the early stages of learning a table, has much to be said for it provided it is not the only method used. Its chief drawbacks are:

1. If persisted in too long it wastes the time of the whole class to give practice to the few.
2. There is a danger that a child will learn some elements in the table incorrectly, thinking, for example, that the class is saying 'Seven eights are sixty-six' instead of 'Seven eights are fifty-six'. Errors of learning of this sort are very difficult to remedy.
3. The response 'Four fives are twenty', say, often cannot be given without the whole of the earlier part of the table being recited.
4. It is difficult sometimes for a teacher to tell whether a child is just opening his mouth and chanting without actual

awareness of what he is saying, or whether he is just repeating what the next child says a fraction of a second later.

5. It assumes that the speed of the chant is the optimum speed for effective fixation of the number facts and that each child works at the same rate.

The following methods of learning tables will be found to give some variety, although it must be remembered that some grouping of children—and very probably some individual work—will be necessary before it can be claimed that the tables are known.

1. Chant tables in chorus and chant backwards.
2. Chant odd (or even) multiples.
3. Establish the meaning of the term *factors* and ask for factors of a given number.
4. Use the division pattern of the tables, e.g. say 'Three sevens are twenty-one', and also 'Three into twenty-one goes seven'.
5. Extend the tables by the addition of a constant, e.g. say 'Three sevens and one are twenty-two'; 'Four sevens and one are twenty-nine', etc.
6. Write a long series of random numbers on the blackboard and multiply them together in pairs.
7. Use the clock-face method, i.e. write the multiplier in the centre of a circle of numbers and multiply them successively by the number in the centre.
8. Use games[1] like *Splash!*, *Fizz-Buzz*, *Lotto*, and so on.
9. The making up and writing out of tables.
10. Display of tables by wall charts.

It is important to keep a record of each child's knowledge of the tables. Each individual pupil should know which tables he knows perfectly, which imperfectly, and which not at all.

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[1] To play *Splash!* a series of circles is drawn across the board to represent stepping-stones, and in each circle a number is written. A child tries to cross the 'stream' by multiplying the numbers in the circles by a given number. When a mistake is made the class cries 'Splash!' Anyone crying out at the wrong time replaces the child making the crossing.

To play *Fizz-Buzz*, an eliminating contest, two numbers—3 and 7, say—are chosen. The class begins to count, replacing multiples of 3 by 'Fizz' and multiples of 7 by 'Buzz'; numbers like 21, 42, etc., are replaced by 'Fizz-Buzz'. Players making mistakes fall out.

Lotto is generally well known.

Written multiplication will begin, in the Infants School, with the writing of the elements of the tables in horizontal and vertical form; thus:

$$7 \times 4 = 28 \qquad\qquad \begin{array}{r} 4 \\ \times\ \underline{7} \\ \underline{\underline{28}} \end{array}$$

and will soon lead on to the use of two-figure multiplicands, like:

$$\begin{array}{r} 21 \\ \times\ \underline{\ 2} \\ \underline{\underline{42}} \end{array} \qquad\qquad \begin{array}{r} 13 \\ \times\ \underline{\ 3} \\ \underline{\underline{39}} \end{array}$$

It is obvious that the multipliers available for use in this step are few in number, for they soon lead to carrying. Provided the tables are thoroughly learnt there is no reason why this step should take very long, for the idea of carrying will have been used already in connection with addition of number. It should be possible, therefore, to proceed fairly soon to the use of multipliers up to 9, which involve the use of carrying one figure.

A child is now ready to proceed to the use of larger multiplicands, but the question of carrying needs careful watching. First examples should be like:

$$\begin{array}{r} 126 \\ \times\ \underline{\ \ 3} \\ \underline{\underline{378}} \end{array}$$

which involves carrying from the units to the tens column but not from the tens to the hundreds column. Following this the carrying should be from the tens to the hundreds column but, when this is first introduced, there should be no carrying from the units to the tens column. Examples will therefore be of the type:

$$\begin{array}{r} 172 \\ \times\ \underline{\ \ 3} \\ \underline{\underline{516}} \end{array}$$

Finally, in this stage the carrying may be both from units to tens and from tens to hundreds columns.

It is now desirable to revise the tens table and multiplication by ten, preparatory to the use of two-figure multipliers. The use of two-figure multipliers immediately brings up the question of whether we should multiply by the right-hand or the left-hand figure first. Most of the arguments in favour of using the left-hand figure first are of doubtful value. There is some point in proceeding in this way if—at a later stage, of course—one is going to teach the multiplication of decimals by the method of arranging the units figure of the multiplier under the last figure of the multiplicand, but otherwise it would appear that the arguments are quite invalid. To state that the first partial product is nearer the answer when the left-hand figure is used seems particularly inappropriate. Surely at this stage the aim is accuracy in arithmetic. And what is the point of it all when the multiplier is a number like nineteen? What is most important is that we should agree on one or other of the methods so that the child is not unduly perplexed when he moves from school to school or from teacher to teacher. It seems that the advocates of the 'left-hand figure first' party hold sway at present. That does not mean, of course, that their method is a better one, since there is no good evidence that one gives better results than the other.

To return to the steps in multiplication, multipliers up to twenty are now introduced, after which larger two-figure, and finally three-figure, multipliers are used. It is desirable eventually to stress the special cases of multiplication by 10, by 100, and by 1,000.

Elementary arithmetic should be taught in a straightforward way and in accordance with common practice. This means that all stunt methods should be avoided—except, perhaps, for end-of-term parties. Occasionally one comes across people advocating such things as *Russian* or *Gypsy* methods of multiplication, involving for small children a most complicated use of the fingers. These should be avoided. Such ideas may be of some use at a later stage in connection with the teaching of algebra, but are quite out of place in the elementary arithmetic course.

Short methods and multiplication by factors should be postponed until a later stage. They are essentially methods for adults. So far as multiplication by factors is concerned, many multipliers do not factorise, and the method has to be adapted to suit their use.

Reviewing the ideas set out in this chapter, it is suggested that the following progressive steps should be used in connection with the teaching of multiplication:

(1) $\begin{array}{r} 7 \\ \times\ 4 \\ \hline 28 \\ \hline\hline \end{array}$ The writing of the elements of multiplication tables in vertical form. After suitable practice, this stage should include the use of noughts in the multiplicand.

(2) $\begin{array}{r} 24 \\ \times\ 2 \\ \hline 48 \\ \hline\hline \end{array}$ The use of two-figure multiplicands without carrying. Again, after suitable practice noughts should be introduced in the units place of the multiplicand.

(3) $\begin{array}{r} 13 \\ \times\ 6 \\ \hline 78 \\ \hline\hline \end{array}$ The use of two-figure multiplicands with carrying. In the first instance 1 only is carried.

(4) $\begin{array}{r} 24 \\ \times\ 6 \\ \hline 144 \\ \hline\hline \end{array}$ As in step (3), but involving the writing of two figures in the last stage of the answer.

(5)	$\begin{array}{r}30\\ \times\ 2\\ \hline 60\\ \hline\hline\end{array}$	Introduction of noughts into the multiplicand.
(6)	$\begin{array}{r}126\\ \times\ 3\\ \hline 378\\ \hline\hline\end{array}$	The use of three-figure multiplicands with carrying from units to tens only. After suitable practice noughts are introduced into the multiplicand.
(7)	$\begin{array}{r}172\\ \times\ 3\\ \hline 516\\ \hline\hline\end{array}$	The use of three-figure multiplicands with carrying from the tens to the hundreds only. As in step (6), noughts are introduced after a suitable period of practice.
(8)	$\begin{array}{r}23\\ \times\ 10\\ \hline 230\\ \hline\hline\end{array}$	Revision of multiplication by 10. Followed by multiplication by 20, 30, etc.
(9)	$\begin{array}{r}26\\ \times\ 17\\ \hline 260\\ 182\\ \hline 442\\ \hline\hline\end{array}$	The use of two-figure multipliers without noughts in either the multiplicand or the multiplier.
(10)	$\begin{array}{r}147\\ \times\ 23\\ \hline 2940\\ 441\\ \hline 3381\\ \hline\hline\end{array}$	The use of three-figure multiplicands without noughts in the first instance, followed by the introduction of noughts.

Finally it may be noted that some of the typical errors made by children in the process of multiplication are:

1. Errors following an imperfect knowledge of tables. Such errors are far more numerous than any other kind, a fact which emphasises the need for accurate and automatic table knowledge.

2. Errors in carrying. A child forgets to carry or carries the wrong figure.

3. Errors arising from poor setting down in which the place value of figures is obscured.

CHAPTER VII

DIVISION OF NUMBER

In elementary arithmetic, division seems to be the process that gives most trouble to teachers. The difficulty is eased to some extent, as indicated earlier, by introducing the notion of sharing; among a group of children pens, marbles, cigarette-cards, and the like are shared.

The first step in formal division is the writing of the tables in division form, thus:

$$\begin{array}{l} 3)\underline{\ 9\ } \\ \ \ \ \ \underline{\ 3\ } \end{array} \qquad \text{or} \qquad \begin{array}{r} 3\ \\ 3\overline{)\,9\ } \end{array}$$

This immediately brings up the question of writing the answer above or below the dividend. In long division the answer is most conveniently written on top, and it would therefore seem sensible to encourage the child to set out his exercises in this way from the start. As against this it is necessary to remember the inherent difficulty of division, and the writing of the answer on top obscures (because of position of hands) part of the exercise being worked; this is not so when the answer is written underneath. If the child is given a suitable time, after having learnt the process of division, to get used to the idea of writing the answer on top, no real difficulty is caused by the change-over.

Following the stage in which tables are written out in division form, there is another when the idea of a remainder is introduced. Exercises should be of the type:

$$\begin{array}{l} 3)\underline{\ 13\ } \\ \ \ \ \ \underline{\ \ \ 4\ }\ \text{and 1 over.} \end{array}$$

The value of learning the extensions of the multiplication tables suggested on page 46 is evident here. The method of writing the remainder calls for some comment. If one were actually sharing instead of dividing figures on paper, one would almost certainly say: 'Four and one over' in the above-mentioned example, and this would appear to be the most sensible way of writing the answer; but obviously there can be no valid objection to *Remainder 1*, *Rem. 1*, or simply *R. 1*. The statement 4 + 1 should *never* be allowed. This has only one meaning, i.e. 5.

From this point the work is carried beyond the tables, first without remainders and later with them. Examples should be of the type:

$$\begin{array}{r} 4\underline{)\ 84} \\ \underline{21} \end{array} \qquad\qquad \begin{array}{r} 4\underline{)\ 87} \\ \underline{21} \end{array} \text{ and 3 over.}$$

These are reckoned outside the tables because the child does not know what four twenty-ones are. So far there have been remainders from the units column only; there have been no other remainders; no 'carrying', as it were. We next proceed to remainders from the tens to the units column but, at the beginning of this stage, omit remainders from the units column. Exercises therefore are of the type:

$$\begin{array}{r} 4\underline{)\ 92} \\ \underline{23} \end{array}$$

Finally, in the short-division stage, there are remainders from both tens and units columns and larger dividends are used.

We turn next to long division. It is sometimes argued that long division should be taught before short division on the ground that short division is long division with the written steps omitted. Indeed, some teachers are very successful in teaching by long division first. But there are difficulties too. The child has to remember where to write the figures, to multiply in the middle of the division exercise, to subtract, and to 'bring down' a figure. It is true, of course, that the child is actually doing these things mentally when doing short division, but he is not conscious of the fact, for he is simply using his knowledge of tables. He is reciting things he has come to know by heart.

In long division it is necessary to analyse the process into its progressive steps, and, so far as possible, teach them one at a time. Obviously in this process the division, multiplication, and consequent subtraction cannot be separated, but they should be taught in a simple unit stripped of all other difficulties.

Less able pupils should start with examples like:

$$\begin{array}{r@{\,}l} & 1 \\ 5) & \overline{5} \\ & 5 \\ \cline{2-2} & - \end{array} \qquad\qquad \begin{array}{r@{\,}l} & 1 \text{ rem. } 2 \\ 5) & \overline{7} \\ & 5 \\ \cline{2-2} & 2 \end{array}$$

and go on to others like:

$$\begin{array}{r@{\,}l} & 5 \text{ rem. } 4 \\ 8) & \overline{44} \\ & 40 \\ \cline{2-2} & 4 \end{array} \qquad\qquad \begin{array}{r@{\,}l} & 8 \text{ rem. } 5 \\ 9) & \overline{77} \\ & 72 \\ \cline{2-2} & 5 \end{array}$$

But there is no reason why the brighter children should not go on, more or less straightway, to examples such as:

$$\begin{array}{r@{\,}l} & 2 \text{ rem. } 1 \\ 21) & \overline{43} \\ & 42 \\ \cline{2-2} & 1 \end{array}$$

These examples are easy since the numbers are small, the multiplication and the subtraction are straightforward, and, in the case of the single-figure divisors, the work is little more than another method of setting down table knowledge. In connection with the earlier work with two-figure divisors, one's choice is limited to some extent, but it has been found in practice that there is ample variety available in the use of 21, 22, 23, 31, 32, 33, 41, 42, 43. The important thing is that the fundamental process of long division is analysed out, and taught as a separate unit devoid of all other difficulties. When this step has been mastered, the size of the dividend may be increased and use made of examples like:

$$\begin{array}{r} 21 \text{ rem. } 2 \\ 21\overline{)\,443} \\ 42 \\ 21\overline{)\,23} \\ 21 \\ \hline 2 \end{array}$$

It has been found helpful to permit a repetition of the divisor as indicated in the initial stages of this step; this is, of course, dropped as soon as the children can do without it.

The next stage is to use divisors like 29, 28, 39, and 38, as approximate answers may be obtained by using 30, 40, etc., as divisors. Miscellaneous divisors and somewhat larger dividends are then introduced. The suggested steps may therefore be set out as follows:

(1) $\begin{array}{r} 3 \\ 3\overline{)9} \end{array}$ Tables written out in division form, without remainders.

(2) $\begin{array}{r} 3 \\ 3\overline{)11} \end{array}$ and 2 over. As step (1) but with remainders.

(3) $\begin{array}{r} 21 \\ 4\overline{)87} \end{array}$ and 3 over. Dividend extended beyond tables. Remainder in units column only. After suitable practice noughts are introduced into the dividend.

(4) $\begin{array}{r} 23 \\ 4\overline{)92} \end{array}$ Remainder from tens column, but not from units. Followed after suitable practice by the introduction of noughts into the units place of the dividend.

(5) $\begin{array}{r} 23 \\ 4\overline{)94} \end{array}$ and 2 over. Remainder from tens column and from units column. Continue practice with occasional noughts in units place.

(6) $\begin{array}{r} 1 \\ 5\overline{)5} \\ 5 \\ \hline \end{array}$ Simple long division. No remainders.

(7)	1 rem. 2 5)7 5 2	As step (6) but with easy remainders. Continue within tables known.
(8)	2 rem. 1 21)43 42 1	Introduction of two-figure divisors within limits suggested in text. Followed after suitable practice with noughts in units place of dividend.
(9)	21 rem. 2 21)443 42 21) 23 21 2	Introduction of three-figure dividends, first without noughts, next with noughts in units place, and finally with noughts in tens place.

(10) More difficult divisors proceeding via 29, 28, 39, 38, 20, 30, 40 to miscellaneous examples.

It is the experience of many teachers that the above-mentioned method works well with quite young children and also with less able children. It should be taught in a straightforward manner and no attention paid to alternatives. This is not to say that other methods are valueless, but simply that confusion follows in the child's mind when he is presented with different methods of doing the same thing. The alternatives may follow at a much later stage in the Secondary School. There is no need whatever to bother about unwieldy examples, and with children of low intelligence it is often impossible to teach long division at all.

It is, perhaps, not out of place to mention other methods of teaching long division here. Among them may be noted: (*a*) the use of factors, and (*b*) the working of a long division sum with a small divisor. Against the first method is the fact that most numbers do not factorise and, too, there is all the difficulty of dealing with the remainders in the intermediate stage, or stages, as the following example shows:

$$4125 \div 32$$

Carry out the division by the factors 8 and 4.

$$\begin{array}{l} 8\overline{)4125} \\ 4\overline{)\ 515}\text{ and 5 over} \\ \quad\ 128\text{ and 3 eights over} \end{array}$$

Thus the remainder is $(3 \times 8) + 5 = 29$.

Against the second method are the facts that there is little point in bothering about a long method when a short one will do equally well and, what is more important from the teaching point of view, the method does not isolate what is fundamentally new in the process.

CHAPTER VIII

UNITS OF MEASUREMENT

THIS chapter is concerned with a most important part of the syllabus, and it is inevitable that more drill will be necessary to learn the relative values of the various units of measurement. A knowledge of these relative values and of the units themselves is of much more importance than the ability to perform lengthy calculations involving them. Every opportunity should be taken to bring children into contact with such units as the pound, the ounce, the pint, the quart, the inch, the foot, the yard, and so on; not necessarily in *standard*[1] measures, but in connection with such things as packets of 'sugar' (sand), lengths of rope, capacity of mugs, milk bottles, and the like. These are the things with which they are in daily contact; the standard measures are in the nature of apparatus for the specialist (see Chapter IV).

Addition and subtraction of money will be discussed at length in this chapter; in so far as it is necessary to teach the rules applied to the other units of measurement, the method will be precisely the same, the only change being the different denominations used. It cannot be over-emphasised, however, that it is our belief that children should not spend an undue amount of time applying the four rules to these latter quantities.

Addition

First steps in the addition of money will be concerned with learning a part of the pence table and the addition of a simple column of pence involving carrying after 12 instead of 10. This is a step where the teacher will find it pays to go slowly. Children do not quickly adapt themselves to carrying after 12 only, when they have accustomed themselves to carrying after

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[1] E.g. copies of the Imperial pound, pint, etc.

10. Particular care will be needed with a column summing to 10*d.* or 11*d.* Before proceeding to examples involving more difficult carrying, it is usually a good thing to introduce a short step involving addition of shillings and pence without the necessity of conversion; this gives the children time to accustom themselves to the appearance of the new sums.

When conversion from pence to shillings[1] has been mastered and the pence table learnt, sums involving conversion from shillings to pounds should be used; at this stage they will not include a pence column and will be of the type:

£	*s.*	*d.*
1	3	0
2	19	0
4	2	0

£	*s.*	*d.*
1	10	0
2	14	0
4	4	0

in order to direct attention to the need for conversion after 20 and not after 10 or 12. Simultaneously with this the shillings table will be learnt. Finally, addition of three columns, involving conversion both from pence to shillings and from shillings to pounds, may be undertaken.

It will be noticed that so far no mention has been made of halfpence. Halfpennies may be postponed until after the final stage of addition of money, if the teacher so desires, or may be dealt with when addition of pence only is being considered.

The steps suggested in this chapter may be briefly summarised as follows:

(1)

s.	*d.*
	8
	2
	10

Addition of pence adding to 10*d.* Followed by addition of pence adding to 11*d.*

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[1] This conversion is done mentally from a knowledge of the pence table. There is no need to carry out division under the answer.

(2)

$$\begin{array}{rr} s. & d. \\ & 9 \\ & 5 \\ \hline 1 & 2 \\ \hline\hline \end{array}$$

Addition of pence involving conversion after 12. Difficulty of sums increased and carrying figure made larger as knowledge of pence table is gained.

(3)

$$\begin{array}{rrr} £ & s. & d. \\ 1 & 7 & 0 \\ 2 & 3 & 0 \\ \hline 3 & 10 & 0 \\ \hline\hline \end{array}$$

Addition of pounds and shillings (or in the case of less able children, shillings only in the first instance), shillings adding to 10. Followed by examples in which the shillings add first to 11, then 12, and finally other numbers up to 19. No figures in pence column at this stage.

(4)

$$\begin{array}{rrr} £ & s. & d \\ 2 & 19 & 0 \\ 1 & 3 & 0 \\ \hline 4 & 2 & 0 \\ \hline\hline \end{array}$$

Addition of pounds and shillings involving simple carrying after 20. Followed by more difficult examples involving larger carrying figures as knowledge of shillings table is gained.

(5)

$$\begin{array}{rrr} £ & s. & d. \\ 2 & 4 & 7 \\ 1 & 17 & 8 \\ \hline 4 & 2 & 3 \\ \hline\hline \end{array}$$

Addition of pounds and shillings involving carrying from pence to shillings and from shillings to pounds.

More difficult examples may now follow as experience is gained.

The following steps are suggested in connection with the teaching of halfpennies:

(1)

$$\begin{array}{rr} d. & d. \\ 2\frac{1}{2} & 4 \\ +\,3 & +\,1\frac{1}{2} \\ \hline 5\frac{1}{2} & 5\frac{1}{2} \\ \hline\hline \end{array}$$

Short stage involving addition of a fraction of a penny without carrying.

(2)

$$\begin{array}{r} d. \\ 2\tfrac{1}{2} \\ 1\tfrac{1}{2} \\ \hline 4 \\ \hline\hline \end{array}$$

Addition of halfpennies involving carrying.

After this halfpennies may be introduced into addition sums from time to time, but care should be taken to see that, if the addition of halfpennies has been taught in the initial stage rather than at the end of the process of addition of money, they do not occur when a new step is being introduced.

Subtraction

As in addition, examples in subtraction should be suitably graded through the series of progressive steps. The only point needing attention at this stage is the method of subtraction to be taught. All the evidence is in favour of the method of equal addition, as in subtraction of number, but it is necessary to think a little more about what is to be said in the actual process of subtraction. Consider the example:

$$\begin{array}{r} 5\,{}^{1}4 \\ -\,2\;9 \\ \hline 2\;5 \\ \hline\hline \end{array}$$

In this the child says: 'Nine from four you cannot. Give it a ten. Nine from *fourteen* leaves five.' And so on. This is possible here because the little 1 written alongside the 4 enables one to regard the upper number as 14 without too severe a tax on the imagination, and because the child has learnt the upper decade number facts. It is not so easy when we turn to money.

In the example:

$$\begin{array}{rr} s. & d. \\ 15 & {}^{12}3 \\ -\;9 & 8 \\ \hline 5 & 7 \\ \hline\hline \end{array}$$

the child says: 'Eight from three you cannot. Give it a twelve.' Now, the presence of the little 12 alongside the 3 does not resemble 15 in the least, and experience goes to show that it is better at this stage to go on to say: 'Eight from twelve leaves four, and three makes seven.' The steps in the teaching of subtraction may be summarised as follows:

(1)

$$\begin{array}{r} d. \\ 11 \\ -5 \\ \hline 6 \\ \hline\hline \end{array}$$

Subtraction of pence only using 10, 11 in the minuend.

(2)

$$\begin{array}{rr} s. & d. \\ 8 & 6 \\ -5 & 4 \\ \hline 3 & 2 \\ \hline\hline \end{array}$$

Subtraction of pence and shillings, without noughts and without need for conversion.

(3)

$$\begin{array}{rr} s. & d. \\ 5 & 6 \\ -3 & 0 \\ \hline 2 & 6 \\ \hline\hline \end{array}$$

As for (2), but involving the introduction of noughts into the subtrahend.

(4)

$$\begin{array}{rr} s. & d. \\ 8 & 0 \\ -3 & 7 \\ \hline 4 & 5 \\ \hline\hline \end{array}$$

As for (2), but involving the introduction of noughts into the minuend.

(5)

$$\begin{array}{rrr} \pounds & s. & d. \\ 5 & 16 & 0 \\ -2 & 9 & 0 \\ \hline 3 & 7 & 0 \\ \hline\hline \end{array}$$

Parallel stage to (2) in pounds and shillings columns. Followed by introduction of noughts first into the subtrahend and then into the minuend.

(6)

$$\begin{array}{rrr} \pounds & s. & d. \\ 9 & 5 & 3 \\ -3 & 16 & 8 \\ \hline 5 & 8 & 7 \\ \hline\hline \end{array}$$

Miscellaneous examples.

As in addition, halfpennies may be introduced immediately after stage (1) or postponed until after stage (6)

Ample practice with the shop method of giving change should be provided, i.e. counting up from the price of the article bought to the amount tendered through a series of short additions using suitable denominations. Cardboard replicas of the usual coins should be used here as before.

In the ordinary written subtractions of money, too, the use of amounts like 2*s*. 6*d*., 10*s*., £1, etc., for the minuend should be emphasised. We need to take odd amounts from such amounts as these far more frequently than we need to take odd amounts from odd amounts.

Other Compound Quantities

As has already been noted, in so far as it is necessary to teach the four rules applied to other compound quantities the methods of teaching and the grading of examples will be similar to those used to teach money. It is our belief, however, that much of the time which was formerly spent in finding the number of inches in a given number of miles, or of multiplying an involved length by an absurdly large multiplier, would be better spent in gaining a first-hand acquaintance with the units themselves. It is seldom that the whole range of units need be taught at any one time. People concerned with heavy goods, for example, seldom worry about ounces. The coal merchant is not summoned for delivering an ounce of coal short. Westaway[1] has suggested that the units should be taught in groups, and experience bears out the value of this suggestion. Suitable combinations of units are:

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[1] Westaway, F. W.: *Craftsmanship in the Teaching of Elementary Mathematics*, p. 48 (Blackie, 1931).

Weight:	ton, cwt., qr.	Used in connection with coal and heavy goods.
	cwt., qr., lb.	Used in connection with wholesale grocery.
	cwt., st., lb.	Used in connection with wholesale meat.
		(The stone is also used, of course, in connection with personal weights.)
	lb., oz.	Used in retail transactions.
Length:	yd., ft., in.	Used by builders.
	pole, yd., ft.	Used by farmers.
	mile, chain, link	Used by surveyors.
	mile, yd.	Used in connection with road distances.

It is suggested that the pole, chain, and link should be postponed until the Secondary School stage, except, perhaps, in those areas where they have a local significance.

Capacity:	gal., qt., pt., gill	Attention should be drawn to the fact that the ratio of the pint to the gill is not constant throughout the country.

Time: year, week, day, hour, minute, second.

In any one exercise, not more than three denominations should be used.

CHAPTER IX

MULTIPLICATION OF UNITS OF MEASUREMENT

In this chapter reference will be made to short and long multiplication, the latter meaning the techniques employed when the multiplier is greater than twelve. As in all other work in arithmetic, the need for multiplication should be demonstrated; e.g. *We need six books each costing 1s. 6d., how much will they cost altogether?* In this chapter only exercises using money will be considered in detail, for the process steps in other quantities follow similar lines.

Short Multiplication

The following examples indicate the main process steps which should be covered by every child:

(1)

$$\begin{array}{lr} & d. \\ & 4 \\ \times & \underline{2} \end{array}$$

Pence multiplied by a number, no conversion to shillings needed.

(2)

$$\begin{array}{lr} s. & d. \\ & 7 \\ \times & \underline{4} \\ \underline{\underline{2}} & \underline{\underline{4}} \\ & 12\overline{)28} \\ & 2s.\ r.\ 4d. \end{array}$$

Pence multiplied by a number, conversion to shillings needed. Note the recommended layout of the working.[1] The child writes in 28 under the bottom line, immediately underneath the pence column, divides by 12 and writes the answer as 2*s*. *r*. 4*d*. The 4 of the answer is placed in the column right underneath the 7 and the 4, while the 2 is placed about one inch to the left.

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[1] If a child has sufficient knowledge of his tables to be able to dispense with these written conversions under the answer line, he should of course be allowed to do so. In multiplication the numbers involved are often much larger than in addition, and the written conversions will be necessary.

$$\begin{array}{lrrl}
(3) & s. & d. & \\
 & 2 & 7 & \\
\times & & 6 & \\ \hline
 & 15 & 6 & \\ \hline\hline
 & 3 & 12\overline{)42} & \\
 & 12 & & 3s.\ r.\ 6d. \\ \hline
 & 15 & & \\ \hline
\end{array}$$

Shillings and pence multiplied by a number requiring conversion from pence to shillings. If a crutch is to be employed place the 3*s*. beneath the answer line and underneath the 2, and place the 12*s*. obtained by multiplying 2 × 6 under the 3. The sum of 3 and 12 is written in the answer line under the 2. This provides good practice which the children will need in long multiplication. Once again see that the shillings and pence are well spaced out.

$$\begin{array}{lrrl}
(4) & s. & d. & \\
 & 1 & 3\tfrac{1}{2} & \\
\times & & 5 & \\ \hline
 & 6 & 5\tfrac{1}{2} & \\ \hline\hline
 & 1 & 2 & \\
 & 5 & 15 & \\ \hline
 & 6 & 12\overline{)17} & \\ \hline
 & & & 1s.\ r.\ 5d. \\
 & & & \\
 & s. & d. & \\
 & 2 & 6\tfrac{1}{2} & \\
\times & & 7 & \\ \hline
 & 17 & 9\tfrac{1}{2} & \\ \hline\hline
\end{array}$$

Shillings and pence with halfpence multiplied by a number requiring the conversion of halfpence to pence and pence to shillings. In this case it is recommended that the conversion from halfpence to pence should be done mentally or in the side margin, and the pence written beneath the answer line as indicated. The writing of a separate column for halfpence is not recommended, as it is divorced from everyday practice. If the child cannot perform the conversion to pence in his head or in a side margin and transfer the result to the answer line, it is doubtful if he is ready for the exercise at all.

$$\begin{array}{lrrr}
(5) & £ & s. & d. \\
 & 1 & 3 & 4 \\
\times & & & 5 \\ \hline
\end{array}$$

Pounds, shillings, and pence requiring conversion from pence to shillings only.

$$\begin{array}{lrrrl} (6) & \text{£} & s. & d. & \\ & 3 & 9 & 8 & \\ \times & & & 4 & \\ \hline & 13 & 18 & 8 & \\ \hline\hline & 1 & 2 & 12)\overline{32} & \\ & 12 & 36 & & 2s.\ r.\ 8d. \\ \hline & 13 & 2\cancel{0})\overline{3\cancel{8}} & & \\ & & \text{£}1\ r.\ 18s. & & \end{array}$$

Pounds, shillings, and pence requiring conversion from pence to shillings and shillings to pounds. The exact layout is shown. The method of dividing by 20 should be explained in an earlier lesson when practising division of number.[1]

Long Multiplication

It is a matter for discussion whether or not children in the Primary School should use multipliers greater than twelve. The view expressed here is that there is no reason whatever why larger multipliers should not be used with the abler children. Where the line of demarcation is to be drawn must be left to the good sense of the teacher. Two methods are generally employed in the teaching of long multiplication to *Junior* School children.

1. The *Wholesale* or *Top-line* method in which the figures of the top line are used as multipliers. The general layout is as follows:

$$\begin{array}{rrrl} \text{£} & s. & d. & \\ 4 & 8 & 7 & \\ & & \times\ 23 & \\ \hline 101 & 17 & 5 & \\ \hline\hline 9 & 13 & 12)\overline{161} & \\ 92 & 184 & & 13s.\ r.\ 5d. \\ \hline 101 & 2\cancel{0})\overline{19\cancel{7}} & & \\ \hline & \text{£}9\ r.\ 17s. & & \end{array}$$

There are a number of points to note:

(*a*) Much preliminary drill is needed in wide spacing and use of figures in the top line as multipliers.

---

[1] Another method equally useful is to cross off the 8 and put it in the answer row, then say: 'Three ten shilling notes equals one pound, which we carry, and ten shillings over.' This one (i.e. the ten-shilling note) is placed to the left of the 8 in the answer row.

(*b*) Each unit is disposed of in turn.

(*c*) This method stresses the fact that 7 × 23 = 23 × 7.

(*d*) The advocates of this method maintain that $\begin{array}{r}23\\ \times 7\\ \hline\end{array}$ is easier than $\begin{array}{r}7\\ \times 23\\ \hline\end{array}$.

2. In this method the top-line figures are not used as multipliers; indeed, the bottom-line number is used as the multiplier in exactly the same way as in short multiplication.

The method of setting down the exercise should be clear from the following example:

$$\begin{array}{rrl}
\pounds & s. & \quad d. \\
4 & 8 & \quad\;\; 7 \\
 & & \;\times 23 \\ \hline
101 & 17 & \quad\;\; 5 \\ \hline\hline
9 & 13 & \;140 \\
80 & 160 & \;\;21 \\
12 & 24 & 12\overline{)161} \\ \hline
101 & 2\cancel{0}\overline{)19\cancel{7}} & \quad 13s.\ r.\ 5d. \\
 & \pounds 9 & r.\ 17s.
\end{array}$$

The pupil can, if necessary, perform the operation 7 × 23 in the margin, e.g.:

$$\begin{array}{r}
7 \\
\times\ 23 \\ \hline
140 \\
21 \\ \hline
161 \\ \hline\hline
\end{array}$$

Those who prefer this method emphasise the points enumerated below:

(*a*) There is no difficulty in multiplying by 23 if the child has been taught the multiplication of number thoroughly.

(*b*) The changing over to the top-line figures as multipliers brings many mistakes.

(*c*) In examples of the type £14 18*s*. 11*d*. × 13 the argument of those who favour the *top-line* method fails to some extent, for it is as easy to multiply by 13 throughout as it is to multiply by 14, 18, and 11 in turn.

Both methods can be made to give good results. Although many opinions have been expressed about the relative values of the two methods, there is no experimental evidence known to us that indicates which is the better method.

Experimental evidence does suggest, however, that with abler children, rather than use the *wholesale* method as such, it is better to use a variation of it in which all the multiplications are carried out first (cf. Bamborough, E. S., 'An Experimental Comparison of Two Methods of Teaching Compound Long Multiplication', *Brit. J. Educ. Psychol.*, 1958, 2: 166–8).

£	*s.*	*d.*
4	8	7
		×23
92	184	161

Application to Other Quantities

All that has been said about process steps in relation to the multiplication of money applies equally well to other quantities. Each child should work through graded process steps which bring out one difficulty at a time, while it may be taken as a rule, to which there are no exceptions, that only three denominations should be handled in any one exercise. Thus after experience of easier examples the child reaches a stage where he can work:

gallons	*quarts*	*pints*
5	2	1
		× 19
106	3	1
11	9	2)19
95	38	9 qt. *r.* 1 pt.
106	4)47	
	11 gal. *r.* 3 qt.	

Reduction from Larger to Smaller Denominations

It is better not to use the word 'reduce' at all, but instead to use the word 'change' or 'bring'. Thus the first exercises in so-called reduction should be oral ones, e.g.:

Bring sixpence to pennies.

How many pennies in one shilling and sixpence?

How many shillings in two pounds?

It can be explained that there comes a time when the examples become too difficult to do in the head, and they must be done on paper. The method recommended is illustrated by the following two exercises:

How many pennies in £2 6*s*. 4*d*.?

$$\begin{array}{rrr}
\text{£} & s. & d. \\
2 & 6 & 4 \\
\times\ \underline{20} & \underline{40} & \underline{552} \\
40s. & 46 & \underline{\underline{556d.}} \\
 & \times\ \underline{12} & \\
 & 552d. &
\end{array}$$

Bring 3 yards 1 foot 8 inches to inches.

$$\begin{array}{rrr}
\textit{yards} & \textit{feet} & \textit{inches} \\
3 & 1 & 8 \\
\times\ \underline{3} & \underline{9} & \underline{120} \\
9\text{ ft.} & 10 & \underline{\underline{128\text{ in.}}} \\
 & \times\ \underline{12} & \\
 & 120\text{ in.} &
\end{array}$$

CHAPTER X

DIVISION OF UNITS OF MEASUREMENT

BEFORE beginning exercises involving the division of units of measurement, the child should be brought to realise the need for such exercises. This may be done by posing such problems as:

If I divide three shillings among six children, how much will each get?

How many lengths of rope each 2 feet long can be cut from a length of rope 30 feet long?

How much money will each of ten children get if 7s. 6d. is shared among them?

How many half-pint bottles of milk can be filled from a can holding a gallon?

Process Steps

In the following steps the divisor will be 12 or less:

(1) Pence divided by a number so that the answer is an exact number of pence.

$$\begin{array}{r} d. \\ 2 \\ 5\overline{)10} \\ 10 \end{array}$$

(2) Pence divided by a number giving an answer in terms of pence and halfpence.

$$\begin{array}{r} d. \\ 2\frac{1}{2} \\ 4\overline{)10} \\ \underline{8} \\ 2 \\ \times 2 \\ 4\overline{)\ 4h.} \\ \underline{4} \end{array}$$

(3)

$$\begin{array}{r} d. \\ 1\frac{1}{2} \\ 3\overline{)4\frac{1}{2}} \\ 3 \\ \hline 1 \\ \times\ 2 \\ \hline 2h. \\ 1 \\ 3\overline{)3h.} \\ 3 \\ \hline \end{array}$$

Pence and halfpence divided by a number giving pence and halfpence.

(4)

$$\begin{array}{rr} s. & d. \\ 2 & 4 \\ \hline 2)4 & 8 \\ 4 & 8 \\ \hline 0 & 0 \end{array}$$

Shillings and pence divided by a number giving no remainder from the shillings.

(5)

$$\begin{array}{rr} s. & d. \\ 1 & 3 \\ \hline 5)6 & 3 \\ 5 & 12 \\ \cline{1-1} 1 & 5\overline{)15} \\ \times\ 12 & 15 \\ \hline 12d. & \end{array}$$

Shillings and pence divided by a number. Be sure that the process of bringing shillings to pence and of transferring the pence to underneath the pence column is grasped.

(6)

$$\begin{array}{rr} s. & d. \\ 2 & 3\frac{1}{2} \\ \hline 3)6 & 10\frac{1}{2} \\ 6 & 9 \\ \hline & 1 \\ & \times\ 2 \\ \cline{2-2} & 2h. \\ & 1 \\ & 3\overline{)3} \\ & 3 \\ \cline{2-2} \end{array}$$

Shillings, pence, and halfpence divided by a number.

(7)	£	*s.*	*d.*	Pounds, shillings, and pence. No remainder from the pounds.
	1	1	11	
	8)8	15	4	
	8	8	84	
		7	8)88	
		× 12	88	
		84*d.*		

(8)	£	*s.*	*d.*	Remainder from the pounds and the shillings.
	1	10	7	
	9)13	15	3	
	9	80	60	
	4	9)95	9)63	
	× 20	90	63	
	80*s.*	5		
		× 12		
		60*d.*		

Short Division

Whenever the child has reached the stage where he can dispense with long division, he should be encouraged to use the short method, as it is so much quicker. Thus at the end of step 6, if the child has become proficient the teacher should make him rework similar examples by the short method. True, he will use crutches at first to help him carry or handle the remainders, but the former will go when the child feels he can do without them.

Division with Divisors Greater than Twelve

Division of money with divisors greater than twelve can be undertaken successfully only with the more able children in the Primary School. With such children, however, when once the steps outlined earlier with smaller divisors have been mastered, further progress may be made. The method of working is exactly the same in principle:

$$
\begin{array}{rrr}
\pounds & s. & d. \\
1 & 7 & 4 \\
\hline
17)23 & 4 & 8 \\
17 & 120 & 60 \\
\overline{6} & 17\overline{)124} & 17\overline{)68} \\
\times\ 20 & 119 & 68 \\
\overline{120}s. & \overline{5} & \overline{} \\
 & \times\ 12 & \\
 & \overline{60}d. &
\end{array}
$$

Reduction from Smaller to Larger Units

The introduction to the method should be through oral questions as:

'How many shillings in twenty-four pence?'

'Change seventy-two inches into feet.'

The teacher can then safely bring in the method to be used when the data contain numbers too large to be handled mentally. The general method is illustrated below:

Change 687 pence into pounds, shillings and pence.

$$
\begin{array}{l}
12\overline{)687d.} \\
2\cancel{0}\overline{)5\cancel{7}s.}\ r.\ 3d. \\
\overline{\pounds 2}\ r.\ 17s.
\end{array}
$$

Quotition

Finally, there is the problem of quotition, where money is divided by money or weight by weight. Again begin the work by oral questioning, e.g.:

'How many threepenny pieces in two shillings and ninepence?'

Later show that a method is needed for a more difficult example, e.g.:

'How many articles each costing 2*s*. 10*d*. could be bought with the sum of £3 5*s*. 8*d*.?'

Bring 2*s*. 10*d*. to pence:

$$\begin{array}{rr} s. & d. \\ 2 & 10 \\ \times\ 12 & 24 \\ \overline{24d.} & \overline{34} \end{array}$$

Bring £3 5*s*. 8*d*. to pence:

$$\begin{array}{rrr} \pounds & s. & d. \\ 3 & 5 & 8 \\ \times\ 20 & 60 & 780 \\ \overline{60s.} & \overline{65} & \overline{788} \\ & \times\ 12 & \\ & \overline{780d.} & \end{array}$$

788 is now divided by 34 in the normal way, giving the answer as 23 articles and 6*d*. remaining over. It is always better to work in usual units such as pence, threepenny pieces, shillings, and so on. If in the above exercise one had been tempted to work in twopences, then 65 would have to be multiplied by 6 and 4 added in, while 2 would have to be multiplied by 6 and 5 added in. We should then have:

$$\begin{array}{r} 23 \\ 17\overline{)394} \\ 34 \\ \overline{54} \\ 51 \\ \overline{3} \end{array}$$

The 3 over would be three twopences. The use of strange units leads to many mistakes.

CHAPTER XI

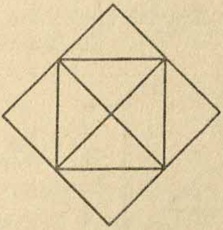
INFORMAL SPATIAL KNOWLEDGE AND SIMPLE PRACTICAL GEOMETRY

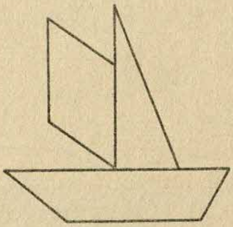
In Nursery and Infant Schools children show a natural interest in shape. Many of the games they play, the toys they use, and the puzzles they employ involve the fitting together and the classification and comparison of shapes of all kinds. They live in a world of shapes and are in constant contact with them.

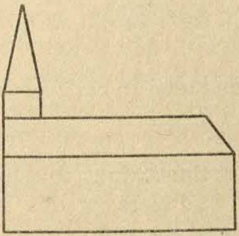
At the other end of school life geometry assumes an importance again. Older children are quick to realise its importance in a scientific and industrial age. It presents constructional problems and provides opportunities for them to work with their hands. In Junior Schools, however, the subject has been neglected. The opportunity has not been taken to link the informal play of small children with the more formal geometry of the Secondary pupils. This, it is believed, is a mistake. Junior children are keen on constructional work; they are for ever making aeroplanes, boats, and the like with paper. They are interested in games of the jigsaw type and they take a delight in making pleasing patterns. A recent move to introduce more geometry into the Junior Schools has resulted in the teaching of the more elementary parts of the Secondary School syllabus, such as, for example, the bisection of lines and angles and the drawing of parallel lines. This seems to us to be an entirely wrong approach, for the Junior School provides an excellent opportunity for the teaching of informal geometry which will be of inestimable value to the child when he arrives at the Secondary School stage and tackles the subject along more formal lines.

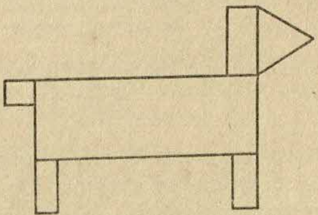
The basic ideas of shape, symmetry, size, and position are quite important in life, and in the same way that the child is introduced to the basic ideas of number by informal play with bricks, marbles, and so on, so he may be introduced to these basic geometrical ideas by informal work with cardboard shapes, by paper folding, by simple measuring, and so on.

The earliest work would probably consist of the making of free patterns with coloured paper shapes and with cardboard tiles. Opportunities will arise in this work to introduce the notion of symmetry. Some of the patterns made will be symmetrical, others asymmetrical. Some will involve a striking departure from the symmetrical. At this stage, too, pictures of shapes, e.g. buildings, gardens, commercial patterns, and so on, may be collected and studied. Exercises in completing symmetrical patterns may be devised. The following diagrams illustrate some of the possibilities:



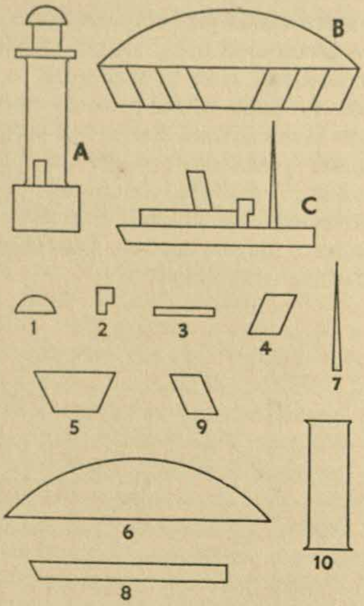






Further work will involve the copying of given patterns and the filling of trays of various shapes. Here the child is called upon to compare and contrast the lengths of the sides of the shapes provided, and to compare and contrast angles. Paper folding and cutting are clearly related to this work.

Following this comes the analysis of patterns into their constituent shapes, which is a useful preparatory stage to more formal work on areas, of which the following are examples:



Which of the shapes 1–10 are included in each of the figures A, B, C?

Patterns may now be made in three dimensions, and involve the use of bricks of various shapes and sizes, prisms, cones, half-cones, hemispheres, pyramids, and the rest.

Simple Practical Geometry and Drawing to Scale

Many teachers in training seem perturbed that they see so little evidence in the Primary School of any work involving simple practical geometry, while practising teachers often seem doubtful of the age at which such work should begin. It is suggested that this side of mathematics must be an integral part of the mathematics teaching in the Primary School. Providing the child has the concept of length, he can be introduced to the foot stick or ruler—exactly 1 foot long and with no divisions—and taught to measure in feet. The inch is then introduced, and a ruler, exactly 1 foot long and divided into inches only, is used to teach the child to measure to the nearest inch. These activities may take place in the top class of the infant school. If so, these simplified rulers should continue in use in the junior school for a while. Indeed, the slower learners may need rulers on which are marked inches, half inches, and quarter inches only for most of the time they are in the junior school. The ordinary ruler, divided into eighths, tenths, and sixteenths of an inch, is too complicated for such children, although children of average and superior learning capacity use it with increasing facility as they move up through the junior school.

The Straight Line

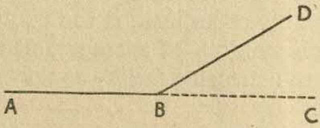
The straight line is best introduced by pointing out the examples that occur in the school and environment, e.g. the edge of a book, the edge of the desk and table, the wall of the school building, the straight line made by the gardener when planting seeds, and the straight road. Further, by using the blackboard and drawing attention to the examples found in everyday life, a distinction can be made between a straight line and a curved line. Before the actual drawing and measuring of straight lines, it is necessary to draw the attention of young children to (*a*) the necessity of clean hands; (*b*) the value of sharp pencils; and (*c*) the correct method of holding the pencil and ruler. As soon as the children begin the actual drawing of the lines, close supervision must be maintained in order to see that they obtain a clear line in which the beginning and ending are clearly indicated. Exercises in the drawing and measuring of straight lines should be graded as follows:

1. Drawing straight lines—no length given.
2. Drawing and measuring in inches.
3. Drawing and measuring in half-inches and quarter-inches.

All three steps can be exercised by drawing simple designs. As soon as a particular child is able, he should be allowed to use the ordinary ruler and draw and measure to eighths or tenths of an inch.

Angles

An angle may be defined for young children as a change in direction. The following example has been found helpful in explaining the definition. A boy is walking along a road ABC.



At B there is a bend in the road and whereas he has been walking in the direction AB he is later walking in the direction BD. The change in direction is indicated by DBC, or the amount of 'turn' between BD and BC and is called an angle. Examples of angles in the classroom should be referred to immediately, e.g. such angles as those formed by opening a door, a book, a desk, or a pair of scissors. The children should then be set to draw angles of different sizes with arms of different lengths (they will not at this stage be able to use protractors).

The children must have their attention drawn to the special angles. Of these the most important at this age is the right angle, and a good way of introducing it is to make use of the children's sense of uprightness. Draw on the blackboard a series of posts some upright and some tilted, or, better still, remind the children of some fencing in the playground, school garden, or near the school, where some posts are tilting over at an angle.

By questioning get the children to point out the difference between the upright posts and those inclined at an angle to the vertical. For the term 'upright' substitute the term 'right angle', and follow up by making the children point out the many examples of the right angle found in the classroom and school buildings. Practice should also be given the children in making right angles by folding newspaper. Finally the 'straight angle' should be mentioned. This may be made clear by indicating that it is formed when two right angles are placed side by side. In our example of the boy walking along the road, the straight angle would be made by a boy who made an 'about turn' and commenced walking in the opposite direction.

The Use of Set-squares and Compasses and the Drawing of the Commoner Figures

Before the child is nine years of age he should be ready to use a set-square in order to draw a straight line at right angles to another. Obviously it does not matter at this stage whether a 60°, 30°, 90° or a 45°, 45°, 90° set-square is used, as the child will be concerned only with the right angle and not with the other angles.

Clear and precise instructions must be given:

1. How to place a set-square and hold it in position.
2. How to draw a line along one edge of the square. The child who has had good practice in drawing straight lines should have no difficulty here.

When announcing the commoner figures to young children there is little advantage in attempting to demonstrate that the figures are plane ones. This may well be left until the child is ten plus or eleven years of age. It will suffice to define the square, rectangle, and triangle as follows:

1. A square is a four-sided figure with all its sides equal and all its angles right angles.
2. A rectangle is a four-sided figure with its opposite sides equal and all its angles right angles.
3. A triangle is a figure of three sides. The right-angled triangle should also be defined.

The circle, however, requires a word of explanation before we can define it simply. The following method of helping children to understand the meaning of the term 'circle' has been proved very successful. Make a chalk mark on the classroom floor or in the playground, and on that spot place a boy, A, holding one end of a short piece of string or rope. Let a boy, B, hold the other end of the rope and, keeping it taut, walk around A, at the same time making a chalk mark on the floor. (Alternatively refer to the example of a bull tethered to an iron stake in a field.) By questioning find out if B moved so that his distance from A was always the same, and so obtain the definition of a circle as a closed curve on which every point is at the same distance from a fixed point called its centre. Explain, too, that the fixed distance is called the radius.

These figures would, of course, be introduced one at a time, and immediately following each introduction the class should be set to construct examples of each type of figure as well as to point out and discuss examples found in the school and classroom. For example, as soon as the children have become acquainted with the square, the teacher should demonstrate on the blackboard, using a blackboard ruler and set-square, exactly how to draw a square. The sequel to this should be the construction on paper by the children of a number of squares of sides of given lengths using ruler and set-square only; the sides should be given first in whole inches and later in inches and fractions of an inch. Similarly with the rectangle.

Before the child can draw a triangle or circle careful instruction must be given in the use of compasses. Using blackboard compasses, demonstrate how to draw a triangle given three sides, and follow by giving exercises for the children to draw for themselves. The teacher must take care that the dimensions given will result in a triangle. The drawing of triangles from data other than three given sides is not recommended in the Junior School. Later, demonstrate the method of drawing a circle and let the children draw circles of varying given radii. In connection with all the work on the construction of the commoner figures, the following points are strongly recommended:

1. Graded work-cards, which have so often been advocated in this book. Have the figures drawn first on sides of a whole number of inches and, later, on sides of inches and parts of an inch.

2. Incorporate the work which has been done on the four named figures into a scheme aimed at the production of various designs. (In connection with this, useful detailed suggestions are given from time to time in certain weekly publications for teachers.)

Finally the child should have the term 'degree' explained to him, and he should be shown how to use a protractor for drawing and measuring angles and for finding out the value of the sum of the three angles of a triangle. Mention must also be made of the term 'perpendicular'.

Drawing to Scale

By nine plus or ten years of age the child should be made familiar with the idea of drawing to scale. Much earlier the child will in some way or other have become acquainted with a map in which a country, county, town, or local area is represented by a picture on a piece of paper. He understands in some way that the map is a picture on paper representing a much larger area on the actual ground. If there is any difficulty here, take the children into the playground, or into a near-by field if possible, and let them draw their picture of the playground or field. Point out at the end that each child's drawing, which we will say is roughly 6 inches by 4 inches, is a picture or 'stands for' the playground or field. Explain then that the representation of a large area or object as a much smaller area or object on a piece of paper is called drawing to scale. The following suggested set of graded exercises in drawing to scale will help to clarify, emphasise, and fix the preliminary explanation. It will be understood that in each case the most suitable scale must be obtained from the children by the teacher.

1. Draw a straight line 6 inches long.

2. Draw a straight line 6 inches long and indicate the inch marks.

3. Take a straight stick 12 inches long on which marks 1 inch apart have been made. Point out that it cannot be drawn in the exercise-book as long as this, and get the class to explain that it will have to be drawn as a line half as long as it actually is. In other words, let 1 inch in the exercise-book stand for, or represent, 2 inches of the real stick. Again, this is drawing to scale. Thus draw a line 6 inches long to represent the stick.

4. Draw the stick again to half-scale, this time indicating the inch marks on the stick. These will, of course, be a half-inch apart on the line in the book.

5. Draw a small book to full scale in the exercise-book.

6. Draw the same book again so that its length is halved and its breadth is halved, i.e. 1 inch in the exercise-book representing 2 inches on the actual book. This process is termed making a plan of the book.

7. Make a plan of a desk top. A scale of 1 inch in the book representing 6 inches on the desk top will be about right.

8. A plan of the teacher's table. A convenient scale here will be 1 inch on the paper representing 1 foot of the table.

9. A plan of the classroom. A suitable scale will be $\frac{1}{10}$ inch on the paper standing for 1 foot of the classroom wall.

10. A plan of the playground. A typical playground might lend itself to a scale of about $\frac{1}{10}$ inch representing 1 yard of playground.

11. A plan of a football pitch (for boys). A scale of 1 inch representing 20 yards is usually suitable.

12. A plan of a netball court (for girls). A scale of 1 inch representing 20 feet is usually suitable.

CHAPTER XII

FRACTIONS

THE range of fractions used in everyday life is small. Few people need more than halves and quarters; even those in specialised callings seldom use fractions beyond $\frac{1}{32}$, with the possible exceptions of $\frac{1}{100}$ and $\frac{1}{1000}$, and in the latter the decimal notation is preferred. There is little justification, therefore, for the complicated manipulations of fractions which formed so large a feature of the older class-books. It is far more important that the idea of a fraction should be grasped than that children should be burdened with lengthy calculations having little or no relation to real life.

The teacher's first task must be to teach what a fraction is; that it is a part of a whole one; that the word *fraction* is virtually synonymous with the words *piece* and *part*; and that pieces or parts are smaller than the whole one. The ideas embodied in the foregoing sentence are of fundamental importance, and must be thoroughly established in the child mind.

A convenient method of doing this is to tear into pieces a sheet of newspaper and to obtain from the class the words *piece* and *part* thereby. A more careful tearing of a second sheet will prepare the way for the establishment of the fact that some fractions have special names, e.g. the half, the quarter, and so on (see Chapter IV).

From this introduction to the idea of a fraction it is possible to proceed more systematically to consider some of the commoner fractions. A circle, for example, may be cut into the various fractional parts required. A single cut through the centre divides the circle into two *halves*. Few children arrive at the age when fractions can be taught without knowing what a half is or how to write the symbol. It is possible to proceed at once, therefore, to cut one of the halves into two quarters and one of the quarters into two eighths. At this point a pause is made in the cutting process to draw attention to the meaning of the fractional notation. It must be insisted that a fraction represents:

$$\frac{\text{The number of pieces}}{\text{The name of the pieces}}.$$

Thus the notation $\frac{1}{8}$ means that the circle is cut into eight equal pieces and that we are concerned with one of them. Similarly $\frac{3}{8}$ means we are concerned with three of the same pieces. Next, proceed to the sixteenth and drive home the point by further examples of the same kind but using sixteenths. Finally, return to the halves and quarters and show that the same idea holds there. It is desirable to proceed as far as the eighth before reference is made to the name of the particular fraction, because from that point onwards—in this particular series of fractions—the naming follows easily from the denominators; this is not so in halves and quarters, and they must therefore be dealt with as special cases after the general nomenclature has been taught.

The lesson on fractional notation is a convenient one for establishing the fact that eight eighths, sixteen sixteenths, four quarters, and so on make a whole one. Examples of the type: *A circle is cut into eight pieces and one is taken; how much is left?* are useful in this connection.

This is also a convenient time to introduce the idea of simple equivalents. The circle diagram readily illustrates the fact that two quarters are the same as a half or two eighths the same as a quarter, and so on. Sooner or later this must be done. A circle or rectangle diagram, suitably divided, readily illustrates. The diagrams should be used at this point to illustrate the principles of cancellation so that a child always expresses a fraction in its lowest terms, or the fraction board can very profitably be introduced at this point. Incidentally, it is worth insisting that the stroke used for cancellation is always made in the same direction.

To fix ideas already taught use simple exercises on fractions of concrete quantities understood by the children, e.g. *What is a half of £1? How much is a quarter of 1s.?*

The Question of L.C.M.

It was always the practice in the past to teach a method of finding lowest common multiples before the four rules applied to fractions, but there would appear to be no justification for this whatever. Calculations involving the addition and subtraction of fractions need only a common denominator to make them possible; the *lowest* common denominator is simply a luxury. Furthermore, in all work designed for Primary School children the figures should be such that a suitable common denominator can be seen at a glance.

It is suggested that for Primary School work halves, quarters, eighths, sixteenths, thirds, sixths, twelfths, fifths, tenths, and hundredths is a sufficiently large range. Inevitably answers will arise in problem work which will introduce fractions outside this range, but so long as the significance of these more difficult fractions is understood, that is all that matters. There is no point in making the work still more difficult by introducing fractions outside the suggested range deliberately.

Addition and Subtraction

In teaching addition and subtraction of fractions—and for much of the way the two operations can be taught at the same time—establish the fact that it is possible to add only things of the same name, for example:

3 potatoes + 2 potatoes = 5 potatoes,
and 3 cabbages + 2 cabbages = 5 cabbages.

But we are unable to do anything with

3 potatoes + 2 cabbages

until we change the name and call them, say, vegetables. And so it is with fractions, e.g.:

1 fifth + 2 fifths = 3 fifths,
and 2 eighths + 3 eighths = 5 eighths,
but, at first, we can do nothing with
1 third + 1 half.

Reference to the circle diagram will quickly establish the equivalence of one-third and two-sixths, and of one-half and three-sixths, so that the last statement may be written:

2 sixths + 3 sixths = 5 sixths.

Proceed immediately to the usual 'trick' for doing this. For example, consider:

$$\tfrac{1}{4} + \tfrac{2}{3}.$$

First decide upon the denomination in which the working is to be done. The product of the denominators already in use is often suitable, as it is here. The sum is written as follows:

$$\begin{aligned} &\tfrac{1}{4} + \tfrac{2}{3} \\ &= \tfrac{}{12} + \tfrac{}{12} \end{aligned}$$

and we say '4 into 12 goes 3, 3 ones are 3'; '3 into 12 goes 4, 4 twos are 8'. The three and the eight are written above the twelves, thus:

$$\begin{aligned} &\tfrac{1}{4} + \tfrac{2}{3} \\ &= \tfrac{3}{12} + \tfrac{8}{12}. \end{aligned}$$

At this stage it is recommended that the denominator be written in twice to provide an opportunity to drive home the equivalence of $\tfrac{1}{4}$ and $\tfrac{3}{12}$ on the one hand and of $\tfrac{2}{3}$ and $\tfrac{8}{12}$ on the other.[1] After the first example or two it is easily seen that it is necessary to write it in once only. To begin with, only pure fractions should be used, and there is no reason why addition and subtraction should not proceed side by side. It will, however, be necessary to keep the two processes separate when mixed numbers are introduced. In examples such as:

$$4\tfrac{3}{8} + 2\tfrac{1}{4}$$

it is best to deal with the whole numbers first and with the fractions afterwards. Begin by adding 4 and 2, and then proceed to add $\tfrac{3}{8}$ and $\tfrac{1}{4}$. The appearance of the work will be:

$$\begin{aligned} &4\tfrac{3}{8} + 2\tfrac{1}{4} \\ &= 6\,\frac{3+2}{8} \\ &= 6\tfrac{5}{8}. \end{aligned}$$

---

[1] It is important to see that the cancelling is well taught at this stage as a preparation for work on multiplication of fractions.

No difficulty arises in the subtraction of mixed numbers so long as the fractional part of the minuend is bigger than that of the subtrahend. When this is not so, two possibilities exist. The subtraction can be carried out either by the method of equal additions or by that of decomposition. Equal additions is a logical sequel to the work in subtraction of number and has much to commend it mathematically. As an illustration of the method consider:

$$5\tfrac{3}{8} - 2\tfrac{1}{2}.$$

This becomes $5\tfrac{11}{8} - 3\tfrac{4}{8}$

$$= 2\tfrac{7}{8}.$$

In subtraction of number the method of equal additions is probably the most effective of those that, at various times, have been in vogue. At least, this seems true for many children. In fractional work, on the other hand—probably because of the added difficulty of changing the whole number into a fraction—it does not seem to be so effective. Experience shows that for this particular operation the method of decomposition is the better.

Consider again the example given above:

$$5\tfrac{3}{8} - 2\tfrac{1}{2}.$$

This becomes $3\,\dfrac{3-4}{8}$

and then $2\,\dfrac{11-4}{8}$

$$= 2\tfrac{7}{8}.$$

Multiplication

Before proceeding with multiplication it is necessary to revise thoroughly the equivalence of fractions.

What always puzzles children at first is the fact that the answer to a multiplication exercise, when a fraction has been used as the multiplier, is smaller than the number started with. This is readily explained by the use of the word *of*.

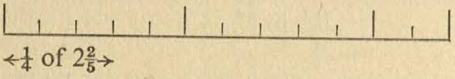
4 × 12 means 4 *of* the groups of 12.

Similarly, $\frac{1}{2} \times 12$ means $\frac{1}{2}$ *of* the group of 12, that is 6.

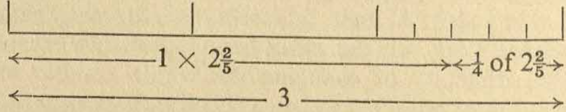
The idea can be substantiated by such considerations as the equivalence of $\frac{1}{2}$ of 5 inches and $\frac{1}{2}$ inch by 5, in which a whole number is multiplied by a fraction. Give ample experience with examples of this type. Now proceed with the use of a diagram[1] considering, say, an example like:

$$1\tfrac{1}{4} \times 2\tfrac{2}{5}.$$

Here is a representation of $2\frac{2}{5}$:



$1\frac{1}{4}$ such groups would be:



$\leftarrow 1 \times 2\frac{2}{5} \rightarrow$
$\leftarrow \frac{1}{4}$ of $2\frac{2}{5} \rightarrow$
$\leftarrow 3 \rightarrow$

Thus we see that the answer to $1\frac{1}{4} \times 2\frac{2}{5}$ must be 3. This is easily obtained if we write the multiplicand and multiplier as improper fractions, and then proceed to multiply the numerators and denominators separately, thus:

$$\tfrac{5}{4} \times \tfrac{12}{5} = \tfrac{60}{20} = 3.$$

No cancelling need be done until the answer is obtained in its first form, but it is customary to cast out the common factors before this stage is reached. The teacher must expect many errors in cancelling in the early stages; there will be cancelling into digits instead of numbers, cancelling one numerator with another and one denominator with another, omitting to cancel because a whole number cannot be cancelled, and so on. Actually, success will be achieved only by considerable practice on the part of the children and vigilance on the part of the teacher.

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[1] The 'proof' of the method of multiplication suggested here will not be readily grasped by all children. Where any difficulty is experienced there is no reason why the teacher should not proceed immediately to multiplying the numerators and denominators separately as indicated later. The multiplying of a whole number by a fraction will have 'justified' this.

It is not recommended that early written work shall be concerned with multiplying a fraction by a whole number. It is better to start with the symmetrical pattern of a fraction by a fraction and proceed by easy stages of cancelling. First examples will be like:

$$\frac{1}{2} \times \frac{2}{5}$$

where a number in the numerator cancels directly with a number in the denominator. The next stage will be:

$$\frac{2}{3} \times \frac{6}{7}$$

where a number in the denominator (or numerator) 'goes into' a number in the numerator (or denominator). From here proceed to examples where numbers in the numerator and denominator have a common factor, such as:

$$\frac{8}{9} \times \frac{7}{12}$$

In this way the child's confidence in himself is established; he meets new difficulties one at a time.

Division

If multiplication of fractions has been well taught, division will give little difficulty. The method of turning the divisor upside down appeals on account of its novelty; it will not, of course, be understood at first. Some justification of the process will be evident when the difficulty of the bigger answer is explained. Introduce this latter by asking *How many twos in eight?* Answer, *Four*. Here we have really divided eight by two. Now ask *How many halves in eight?* Answer, *Sixteen*. Here we have divided eight by half. In the exercise:

$$8 \div \frac{1}{2}$$

we get the same result if we turn the divisor upside down and multiply, thus:

$$8 \times \frac{2}{1}.$$

Many simple mental problems of the type: *How many half-inches in five inches?* should now be used to drive home the process.

Finally, an explanation of the reason for inverting the divisor should be given. It will not be understood by all at first, and, indeed, some will never understand it.

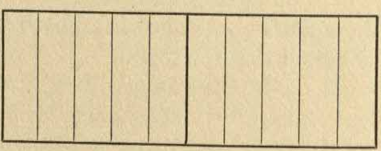
Consider an example like:

$$7\tfrac{1}{2} \div 2\tfrac{2}{5}$$

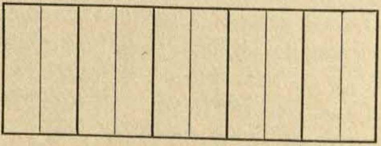
which is worked in the form:

$$\tfrac{15}{2} \times \tfrac{5}{12}$$

The problem at first means $\frac{15 \text{ halves}}{12 \text{ fifths}}$. In order to carry out this division it is necessary to change both the halves and the fifths into fractions of the same name, here tenths. To turn halves into tenths each half must be cut into five pieces, as the following diagram shows:



and to turn fifths to tenths we must cut each into two pieces, as in the diagram:



Thus we have $\frac{15 \times 5}{12 \times 2}$, which is the same thing as $\frac{15}{2} \times \frac{5}{12}$

Here it is evident that the original dividend is followed by the divisor turned upside down with the division sign changed to the multiplication. The inversion of the divisor is thus seen to be a device for changing the numerator and denominator to fractions of the same name. In practice it will be found desirable to use diagrams to elucidate this point as suggested; otherwise an impression is easily given that the value of the fraction is being altered.

The following, then, are suggested steps in the teaching of fractions:

1. Fractional notation; simple equivalence of fractions.
2. Addition and subtraction of two simple fractions, e.g. $\frac{1}{2} + \frac{1}{3}$, $\frac{3}{8} - \frac{1}{4}$.
3. Addition and subtraction of mixed numbers like $4\frac{3}{8} + 2\frac{1}{4}$, $3\frac{5}{6} - 1\frac{1}{3}$, but avoiding examples such as $4\frac{1}{6} - 1\frac{7}{8}$.
4. Subtraction. Examples such as $4\frac{1}{6} - 1\frac{7}{8}$.
5. Multiplication. Examples such as $\frac{1}{2} \times \frac{2}{5}$, in which the same figure appears in both numerator and denominator.
6. Multiplication. Examples such as $\frac{2}{3} \times \frac{6}{7}$, in which a number in the numerator (or denominator) is a simple multiple of a number in the denominator (or numerator).
7. Multiplication. Examples such as $\frac{8}{9} \times \frac{7}{12}$, in which common factors are cancelled.
8. Multiplication. Examples such as $1\frac{5}{6} \times 1\frac{1}{5}$, which involve mixed numbers.
9. Division. When the principle of inverting the divisor has been taught, there is no point in precise grading of examples at this stage as adequate experience of multiplication of fractions has been gained.

An understanding of fractions, other than $\frac{1}{2}$ and $\frac{1}{4}$, is difficult for many children even at 9 years of age.[1]

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[1] See Piaget, J., Inhelder, B., and Szeminska, A.: *The Child's Conception of Geometry* (Routledge & Kegan Paul, 1960). Also, Lovell, K., Healey, D., and Rowland, A. D.: 'The Growth of Some Geometrical Concepts', *Child Development*, 1962, 33.

CHAPTER XIII

DECIMALS

It is far more important that children should have a thorough understanding of the decimal system of notation than that they should be able to perform intricate calculations with decimal quantities. And it is not necessary, for most people at any rate, that their knowledge of the system should extend from millions to millionths. In all work in arithmetic, attempts should be made to introduce a topic in a real-life way. With decimals this may be done by using the various forms of meters which have decimal notation. A type that most children are familiar with nowadays is that part of the speedometer of a car or motor-cycle which registers the number of miles covered. Such an instrument on a new vehicle would appear:

0	0	0	0	0	0

After running one mile it would register:

0	0	0	0	1	0

and after nine miles:

0	0	0	0	9	0

A further mile's running would bring a one into the second column and the nine would be replaced by a nought, thus:

0	0	0	1	0	0

and so on. This sort of thing may be continued, using only the whole numbers and avoiding the decimals, for so long as the teacher feels it necessary with his particular class. It is then pointed out that a mile is quite a long way, and there is no reason why the same system should not be used for measuring parts of a mile, so long as the figure beyond the units figure is distinguished in some way from the rest. On a mileometer this figure is usually red in colour, but any other distinguishing device would do just as well.

When writing such figures on paper, we insert a dot (called a decimal point) between the units figure and the figure which is measuring parts of a mile. Since this last column on the mileometer has ten figures like all other columns, it is obviously measuring tenths of a mile. Thus the reading:

0	0	5	6	7	8

records the fact that the car has travelled 567·8 miles or 567 and eight-tenths miles. Even a tenth of a mile is a long way, viz. 176 yards, and there is no reason why another figure should not be introduced measuring tenths of tenths of miles (i.e. hundredths of miles). This may be done on the blackboard, although it is not common practice to have such a figure on a speedometer. In this way the decimal system of notation is given a real meaning, and the relationship between the decimal and vulgar notations is easily established. It is soon seen to be but an extension of the ordinary system of numbers.

Ruler work, using tenths, is useful in connection with introductory work on decimals, but it must be noted that some children are able to cope with decimals on paper, but are not yet ready to handle a ruler with any degree of accuracy because of insufficient experience of drawing and measuring lines. The difficulty may, however, be overcome by means of a well-graded set of exercises. At first the children will measure lines the lengths of which are made up of whole numbers of inches, and will go on to draw lines of similar lengths. Such work will be followed by the measuring of lines made up of inches and tenths of inches, e.g. 3·4 inches, 4·7 inches, so that the need for some unit smaller than an inch becomes apparent. It is easily seen that the ·4 and the ·7, in the examples shown, are convenient ways of writing down the fact that the lines measured were four small divisions, i.e. $\frac{4}{10}$ inch, and seven small divisions, i.e. $\frac{7}{10}$ inch, respectively, beyond the last inch-mark used. As in the previous step, the work on measuring lines of such lengths is followed by the drawing of lines of similar lengths.

Addition and Subtraction of Decimals

Once the system of notation has been grasped (and even before it is grasped!), addition and subtraction of decimals present little difficulty. There is nothing new to learn except the keeping of the figures in columns. This of course has always been necessary, but the presence of the decimal point has an uncanny knack of causing trouble here. The use of noughts in the blank spaces is often useful in overcoming this difficulty.

One method of giving experience in reading, writing, and setting down decimals is to divide the blackboard into columns headed thousands, hundreds, tens, units, tenths, and hundredths, thus:

Th	H	T	U	*tenths*	*hundths.*
	1	2	4	3	1
1	0	1	4	2	
			1	7	
				etc.	

and to require the children first to copy, and later to write from dictation, numbers against this background.

As a further safeguard against confusing the columns, an introductory exercise might well consist simply of the setting down of an addition exercise without going on to do the actual process of addition.

Multiplication of Decimals

This process calls for more discussion. There are several methods that might be used, but one, viz. standard form, may be dismissed at the outset as being an unnecessary encumbrance. There are just one or two places in mathematics where a knowledge of standard form is useful, e.g. in contracted methods and in the astronomical and physical applications of mathematics. Those who need such knowledge may be left to acquire it at the appropriate time. Most people will get along quite happily without it. The two most popular methods are the *traditional* and the *modern* methods. In the traditional method the decimal points are ignored until the multiplication has been carried out. The number of digits in decimal places in the multiplicand and multiplier are then added, and the decimal point inserted in the product to coincide with this number. Consider the example:

$$12{\cdot}367 \times 2{\cdot}34.$$

The multiplication is first carried out as though no decimals occurred, thus:

$$\begin{array}{r} 12367 \\ \times\ 234 \\ \hline 24734 \\ 37101 \\ 49468 \\ \hline 2893878 \\ \hline\hline \end{array}$$

The answer to the original sum is now obtained by counting the number of digits to the right of the decimal points in the multiplicand and multiplier, i.e. three in the former and two in the latter, making a total of five, and allowing for this number of places in the product obtained. Thus:

$$12{\cdot}367 \times 2{\cdot}34 = 28{\cdot}93878.$$

The method of arriving at the position of the decimal point is easily explained as follows:

$$\tfrac{1}{10} \times \tfrac{1}{10} = \tfrac{1}{100}$$

or, in decimal notation, $\cdot 1 \times \cdot 1 = \cdot 01$, i.e. two places of decimals in the multiplicand and multiplier result in two places of decimals in the product. A few more examples of this kind, avoiding of course in class the use of so clumsy a term as multiplicand, soon justifies the method. The method is easy to teach and easy to remember and, provided the tables are known, it is mechanically safe. It was formerly regarded as unmathematical because the actual multiplication was carried out with the decimal points ignored; this resulted in the partial products being untrue. Set out as suggested above, however, this criticism is no longer justified. After considerable experience in teaching both this and the modern methods, and after having started teaching with a bias towards the modern method, we recommend this traditional method as being more satisfactory for teaching-purposes in Primary Schools.

The modern method is to set out the example so that the units figure of the multiplier is written under the last figure of the multiplicand. Since a decimal number multiplied by a number less than ten but not less than one remains in the same relationship to the decimal point, the position of the point in the multiplicand is unaffected throughout the process and each partial product is mathematically correct. The example worked above would appear as follows when worked by the modern method:

$$\begin{array}{r@{\cdot}l} 12 & 367 \\ \times\ 2 & 34 \\ \hline 24 & 734 \\ 3 & 7101 \\ & 49468 \\ \hline 28 & 93878 \\ \hline \end{array}$$

From the mathematical point of view this method has much to commend it. The decimal points, from that in the multiplicand, remain in a vertical line, and the figure obtained as a product at each step of the multiplication is written under the figure in the multiplier being used. This is precisely where the difficulty arises in teaching the method; for the decimal point in the multiplier takes up some room, and children are apt to give it more than it needs and thus get an extra column of figures in their working. The method also leads to many mistakes when the multiplier is a decimal fraction. Incidentally, this is the only occasion in school arithmetic known to the writers when it is of some advantage to multiply by the left-hand figure first; and even here it is not absolutely necessary.

Remembering that children are fairly experienced multipliers before they start on work with decimals, the following progressive steps are suggested for this work:

(1) 13·6 × 7 Decimal figure in multiplicand only; single-figure multiplier; no noughts.

(2) 20·7 × 6	Introduction of noughts into multiplicand.
(3) 14·3 × 2·3	One digit in decimal place in multiplier. No noughts.
(4) 30·4 × 2·6	Introduction of noughts into multiplicand.
(5) 45·86 × 3·4	Larger multiplicands, at first in the direction of the decimal figures and afterwards in the direction of the whole numbers.
(6) 34·78 × 2·3	
(7) 45·3 × 1·06	Introduction of noughts in the middle of figures constituting the multiplier.
(8) 34·2 × 0·14	Multiplication by decimal numbers only.

Division

Since children will not tackle division of decimals until they have mastered division of number in the ordinary way, they seldom have much difficulty in learning the little extra that is needed to perform the operation. By far the best method is that which makes the divisor a whole number, with a corresponding adjustment in the dividend. The method is justified by considering an example like:

$$\begin{array}{r} 4 \\ 3\overline{)12} \end{array}$$

Provided the divisor and the dividend are both multiplied or divided by the same number, the quotient remains unaffected. Several examples should be used to demonstrate the truth of this. *Moving a decimal point*[1] is simply multiplying (or dividing) by 10, 100, etc., and so long as this is done to both the divisor and the dividend, the result remains unchanged. It is necessary to insert the decimal point only when it is passed in bringing figures down from the dividend. Attention should be drawn to the fact that in this method the remainder is not accurately obtained.[1] But surely the whole point of using decimal division is to obtain as accurate a quotient as possible and to render the remainder insignificant. From a mathematical point of view, then, it is necessary only to remember that the remainder is not accurate, except in those cases where no movement of the decimal point was necessary.

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[1] Strictly speaking, it is the figures which move and not the point.

The following steps are suitable for the teaching of this process:

(1)	3)34·6	Only whole numbers in divisor. No noughts.
(2)	4)50·8	Introduction of noughts into dividend.
(3)	3·4)12·46	One decimal place in divisor; two decimal places in dividend so that a decimal figure remains after moving point.
(4)	4·6)68·02	Introduction of noughts.
(5)	5·67)45·867	Larger divisors and correspondingly larger dividends.

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[1] Unless expressed as a fraction.

CHAPTER XIV

THE MENSURATION OF RECTANGLES AND CUBOIDS

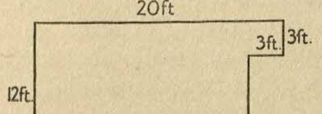
The Idea of Area

AN excellent way of broaching the topic is to ask the children how they would find out if one piece of wood was longer than another, or a desk longer than a table, or one classroom longer than another (see that the work of Chapter XI has been thoroughly covered). Again, question the class concerning the units which would be used in these and similar activities. The object of this questioning is, of course, to make it clear that if we wish to find out if one thing is longer than another, we find the length of each in inches, feet, or yards, as the case may be. Next hold up two books which are almost but not quite the same size, and ask how we could find out which of the two was the larger. Or ask how we could find out which classroom in the school had the largest floor space. To show that just as there are units known as inches with which we measure length, so we have a square inch for measuring areas. Thus the area of anything may be defined as its measurement in square units. Give the children experience of drawing a square inch on paper, and in drawing a square foot and square yard in the playground, using chalk.

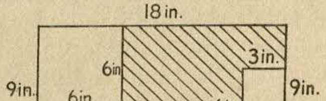
Areas of Squares and Rectangles

Divide the class up into about five groups, and let each group draw a square or rectangle using suitable dimensions. For example, figures with dimensions (all in inches) 3 × 2, 4 × 3, 3 × 3, 4 × 2, and 5 × 2 would do very well. In addition, give out to each child a dozen or so squares each of area one square inch; these can easily and quickly be cut from newspaper or other waste paper using a guillotine which is used in craft lessons. Next show the children how to divide their figures into squares by dividing the sides of their square or rectangle into inches and joining up opposite division marks. Finally, tell the children to place the pieces of paper each one square inch in area over the squares which have been formed in their figures, and to count up the number of square inches of paper required to cover their figure. Ask each group for the area of their figure and how they obtained it; then point out that the area of every square or rectangle could be obtained by counting squares, but that such a method would be long and tedious. By further questioning draw from the class the fact that the area is equal to the number of units of length multiplied by the number of the same units of breadth; here, therefore, the formula for the area of a square or rectangle is length × breadth.

Having found the expression for the area of a square or rectangle, follow up by giving as much experience as possible in drawing and measuring squares and rectangles on paper and calculating their areas, and in measuring and calculating the areas of classroom floors, the playground, and patches of the school garden. Insist on the correct units being used (i.e. square inches, square feet, or square yards). This in turn should be followed by mental work (of the type 4 yards by 16 yards, 8 feet by 5 feet), and finally exercises of the kind illustrated below should be set for the more able children.

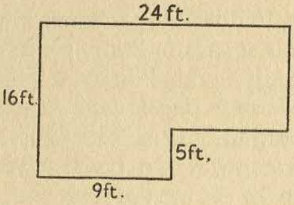


Find the area of the shape.

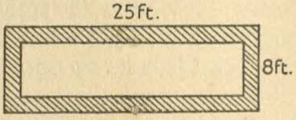


Find the area of the shaded shape.

Such examples can sometimes be obtained by measuring parts of the school buildings or grounds; if not, use the textbook exercises or make up suitable ones on work-cards.



Find the area of the shape.

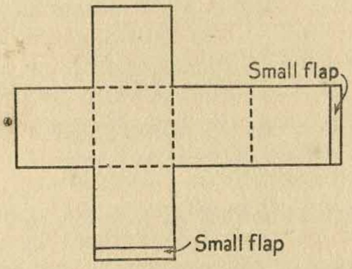


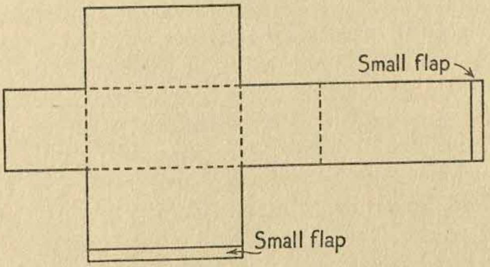
If the border shaded is 2 feet wide, find the area of the inside piece.

Volume of Cube and Cuboid

First explain the need for units of volume. This can be done by demonstrating the need for a unit so that one can find out which of two boxes is the larger. That is to say, we wish to find out which of the two boxes has the greater size or occupies the greater amount of space. We then have to show that the volume of a cuboid is equal to the number of units of length multiplied by the number of the same units of breadth multiplied by the number of the same kind of units of height. To do this it is strongly recommended that a number of wooden cubes each one cubic inch in volume be purchased. The expense involved is very little when it is remembered that the cubes will last a lifetime. From sixty-four of such cubes one can build up a considerable number of cuboids of different volumes. Making the solids should be done by small groups of children, each group in turn counting up the number of small cubes used, measuring the external dimensions of the completed solid and verifying that the number of one-inch cubes used is equal to the length of the cuboid, multiplied by its breadth, multiplied by its height when each dimension is measured in inches. Follow up by as much practical work as possible involving the measuring up and calculating the volumes of boxes and classrooms, and finally proceed to mental exercises.

A useful task for children is to make a cube or cuboid from the corresponding development. The development is drawn on





paper to given measurements, the paper is bent along the dotted lines (obviously the development has been cut from the paper), and a solid is made by adding a touch of glue or paste to the small flaps as indicated in the diagrams above.

In connection with this chapter and Chapter XI, readers should consult Piaget, Inhelder, Initial and Szeminska, *op. cit.*; also Lovell, Healey, and Rowland, *op. cit.*

CHAPTER XV

PRINCIPLES TO BE OBSERVED IN PROBLEM WORK

Structure of Human Abilities in Relation to Performance in Arithmetic

As long ago as 1904, Charles Spearman first advanced the view,[1] based on his own researches involving sensory tests and estimates of intelligence, that there was one fundamental function underlying all intellectual activities, whereas the remaining abilities in each activity seemed to be different from that in all the others. He and his students concluded, therefore, in the above-mentioned and in later researches, that every task involving intellectual activity was dependent upon a general ability which he called 'g' and a separate ability specific to each task involved. Spearman refused to identify 'g' with intelligence, but suggested that it was the general mental energy with which each individual is endowed. Nevertheless, by studying tests which are highly saturated with 'g', psychologists have now reached a measure of agreement that a useful working definition of intelligence is:

1. The ability to see relevant relationships between objects or ideas; and

2. The ability to apply these relationships to new but similar situations.

Later workers, such as Burt, Stephenson, and El Koussy, claimed that they had established the presence of other abilities independent of, and over and above, general intelligence, which made important contributions to scholastic performance. Of these the most important was 'v', the ability to handle words and verbal relationships generally, and 'k', an ability of a spatial perceptual nature. Work during the 1939–45 war with large groups of testees showed that in a general population (i.e. one such as would be obtained by taking 1,000 men at random off the streets of a large city), over and above general intelligence there may be said to be a verbal-numerical-educational ability on the one hand (known as the 'v:ed' factor) and a practical-mechanical-spatial-physical ability (known as the 'k:m' factor) on the other. From our point of view only the 'v:ed' factor is of importance, since there is no evidence that the 'k:m' factor contributes in any way to performance at arithmetic at the junior school stage.

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[1] Spearman, C.: 'General Intelligence Objectively Determined and Measured', *American Journal of Psychology*, Vol. 15, 1904, pp. 201–93.

Vernon,[1] quoting results which he obtained from tests given to 1,000 Army recruits (a fairly unselected sample), indicates that in tests of arithmetic involving mechanical and problem work about fifty-eight per cent of the variability is attributable to 'g' (general intelligence) and some twenty-six per cent due to 'v:ed'. Thus in tests of arithmetic it appears that 'g' is rather more than twice as important as the 'v:ed' factor. Again, both Service psychologists and those working with children have clearly established that 'g' contributes more to performance at problem arithmetic than purely mechanical arithmetic. It is, of course, to be expected for such a problem as:

Fifteen shillings is to be divided among three boys so that the second gets twice as much as the first and the third half as much again as the second. How much does each get?

demands more intellectual power than multiplying 276 by 9. Hence it may be said:

1. Performance both at problem and at mechanical arithmetic depends on 'g' and 'v:ed'.

2. Since 'g' is responsible for by far the greater part of the variability of test performance in arithmetic, and problem work depends more on 'g' than mechanical arithmetic does, it follows:

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[1] Vernon, P. E.: *Structure of Human Abilities* (Methuen), 1st edition, 1950, p. 23. This book gives a clear and concise account of recent research into the problem of human abilities.

(*a*) The less able children will do relatively better at mechanical arithmetic as compared with problem work.

(*b*) Children are likely to show more unevenness or specific backwardness in simple mechanical arithmetic than in problem arithmetic; for problem arithmetic—depending more on 'g'—will be associated with a generally smoother and higher level of all-round performance.

Types of Problem

Some teachers in the past thought only of problems that were to be found in textbooks, whereas in real life problems more often involve immediate aspects of the environment than other people's descriptions of them. It is, of course, very important that children should be able to solve textbook problems, but many tasks involving weighing, measuring, constructing, and calculating can be, and should be, found in the environment. For example, to find the postage on a parcel, the child has to weigh it, find the postage rates, and calculate the postage.

A most important point to remember, regardless of the type of problem, is that mechanical arithmetic and relevant problems should always be worked together. The need to work mechanical exercises grows out of the need to work everyday problems.

The Progressive Steps to be Observed in Helping a Child to Solve Problems

From what has been said in the previous paragraphs it will be clear that problem work will always tend to be more difficult than mechanical arithmetic, and so every teacher should introduce problem work as early as possible (in the Infant School) and proceed by progressive steps. If a child reaches the Secondary School and cannot 'do problems', nothing can be done overnight to remedy the situation. The time to begin problem work is when the child is six or seven years of age. Thus the following points are strongly recommended:

1. As soon as the child can manipulate the four rules in number (totals, products, and dividends not to exceed, say,

thirty) and can read, then give orally, and write up on the blackboard, simple problems illustrating the four rules. Call the work *Puzzles*. For example, in addition:

Put eight beans in one jar and four beans in another. How many beans in both jars?

The child puts out the beans as directed, adds them up and makes the sum, $8 + 4 = 12$.

In division one might have:

You have eighteen beans. If you put them into three jars, so that there are the same number in each, how many will there be in each?

The child puts out the beans into the jars (shares), then works the exercise on paper, $18 \div 3 = 6$.

At the very beginning, and from time to time throughout the teaching, see that *verbal and written statements are linked with objective demonstrations*. Thus, at any time in the Primary School, revert to the use of apparatus, practical experiment, and objective demonstration when a new rule or technique is being embodied in problem form. Then, and then only, let the children use work-cards on which the problems alone are written (work-cards here includes textbooks). It is also wise to work problems on each of the four rules in turn, in the order addition, subtraction, multiplication, division, before cards are given on which the rules are scrambled. Also see that the problems are well graded; one-step problems to begin with, then two-step problems, and so on.

2. As in the Infant School, number was introduced through the use of concrete apparatus, so at each stage in the Junior School the written problem must be in terms of real life, e.g. sharing money, size of room, distance travelled by motor-car, weight of coal bought. It is true that many textbooks did not in the past provide real, meaningful problems, but the position is rapidly improving. Nevertheless, more textbook analysis and teacher criticism of textbooks, such as has been undertaken in America, could be most usefully undertaken in this country.

3. The language used in problems must always be simple,

and the data clearly presented and free from all ambiguity. Children sometimes fail at problems because they cannot realise the exact situations that the words describe. Problem work in arithmetic is not to be regarded as a test of verbal ability, but as an exercise in applying mechanical arithmetic to the type of problem the child will later meet in real life.

4. All problems, as they are introduced after some new piece of work has been taught, must be easy to begin with, and progress by easy steps so that the pupils can tackle the problems unaided with a fair chance of success. It is useless giving help with problems, other than the slightest of hints, for once the method is indicated it is no longer a problem to the child.

5. It is absolutely essential for the child to get the full meaning of the problem every time. Hence give the children explicit training in asking themselves: 'Is adding, subtracting, multiplying, or dividing called for?'; 'Is more than one of these processes involved?'; 'What are we told?'; 'What are we asked to find?'; 'What units do we work in?'; 'Do we change the units in the working?' And teach one rule—never to be varied—always to draw a diagram to illustrate the problem if possible.

Meaningful Problems

There is no doubt that problems based directly on their experience, or based on a setting which they well understand and can picture, are the best ones for children. Furthermore, as adults many will limit their arithmetic to the needs of real life. Again, school life is short, money for educational purposes limited, the children and their teachers live in a real world, not Utopia, and it is not possible to take the children everywhere and get all the experiences that some educationists would demand. Thus problems cannot be confined to the poultry which the children keep and the rail and incidental expenses which were incurred on a school journey. Rather there must be a variety of problems covering all the main phases of the work. The important thing is that the problems should be real, meaningful ones about things in which the children are interested, even if the actual setting is not always a familiar one. Hence there should be problems concerning the games the children play, the classroom shop, real shopping outside the school, school lunches and savings, post office transactions, weighing and measuring, speeds of cars, trains, and ships, paper and cardboard construction work, and so forth. As already stated, it is hoped that teachers will keep a close watch on textbooks in this respect.

While British psychologists have been very industrious in studying the structure of abilities in relation to performance in arithmetic, it is Jean Piaget who in recent years has given penetrating insights into the manner in which children form concepts of quantity, space, weight, volume, number, time, movement, and speed. The experimental population consisted of his own children and those attending the Geneva schools. Experiments were performed in front of children using familiar material like counters and modelling clay, and they had to predict their outcome. Thus, in studying the development of the concept of quantity, two equal-sized balls made of modelling clay might be shown to the child, both of which he recognises as containing the same amount of material. One of the balls is then altered in shape to become that of, say, a sausage. At first the child will not admit that the two objects contain the same amount of clay; that is, he denies conservation of quantity in the example. When rather older he will admit it in some cases but not in others; later still he will agree to the conservation of quantity in all cases. In the first two stages his thinking seems to be affected by cross-section, shape, or thickness, and the concept of conservation of quantity is only formed when the child realises that the clay sausage can be returned to its original shape. For Piaget this 'reversibility' is a fundamental characteristic of thought, and is essential for any form of mental experimentation and for logical inference. He claims that the concept of quantity appears on the average between 7 and 8, that of weight between 9 and 10, and that of volume between 11 and 12 years of age. In his experiments on the development of the concept of number, a child, say five years of age, might be asked to put out as many red counters as there are blue counters. If the counters forming one of the rows are now spread a little more, he believes that the longer row contains more counters. Number and space at that age seem to form a perceptual whole. By about seven years of age, on the average, the notion of the conservation of number has been formed, and if the intervals between the counters in the rows are now varied, the child now no longer maintains that there is a change in the number of counters. He now seems to be aware that the increase in the length of the row is proportional to the increase in interval between the counters. Piaget's work suggests, then, that the notions of number, quantity, space, time, and so forth are not innately given intuitive capacities but are built up bit by bit. (See also Bibliography.)

CHAPTER XVI

THE VALUE OF MENTAL WORK AND THE STANDARDISATION OF PRIMARY SCHOOL METHODS. REVISION. ASSESSMENT OF PROGRESS

The Value of Mental Work

ULTIMATELY all arithmetic is mental. Number relationships are carried out in the mind, and the discernment of processes to be used in the solution of arithmetical problems are mental operations. We resort to writing only when the figures involved are too large or the processes too long to be carried out easily in the mind. The work on paper is really in the nature of jottings to aid the memory and to relieve the mind of the burden of factual data while it is concerned with the unravelling of the arithmetical problems themselves. Thus, whenever we do arithmetic we are doing mental arithmetic. Nevertheless, when we speak of mental arithmetic in connection with school work, we have a somewhat different meaning of the term in mind. Mental work in school is concerned with calculations which are both simple and short enough to be performed entirely in the mind and without jotting on paper at all. Actually there is no reason why we should be so rigid in our mental work as this; the child might well be permitted to make such jottings as are necessary to enable him to get his sums right. Getting the right answer is the goal of all arithmetic, and any props which a child needs to enable him to reach this end should be permitted. He should of course be encouraged to dispense with such props as soon as he can manage without them.

Mental arithmetic is valuable because it gives practice in the use of the simpler number relationships so that they become automatic. It is thus quite a good plan to include one or two of these simpler relationships in every set of mental sums. Mental arithmetic, too, provides opportunities for computation by short methods and by 'adult' methods. This means that children must be taught to think in terms of numbers rather than figures. An adult confronted with 18 × 17 does not visualise the two numbers set out as in ordinary multiplication. He probably carries out the multiplication by adding 180, 70, and 56. This follows naturally from a knowledge of the fact that 18 is 10 plus 8, and that 17 is 10 plus 7.

Another important use of mental work is that it enables a teacher to revise new rules quickly. Mental problems set on a new rule would necessarily be simple, but they serve to establish the rule and give an indication of how well it is understood. Such mental work gives an opportunity, too, to test the ability of a class to apply a new rule.

Finally, mental work enables a teacher to keep revision up to date. When no particular rule has to be revised, the opportunity may be seized to revise a whole range of topics by means of a set of simple mental exercises.

It has been found useful as a teaching device to write on the blackboard, perhaps in note form, the essentials of a mental problem. For example, if the problem is *How much will each child receive if 3s. 4d. is shared between 10 children?*, a note on the board something like *3s. 4d.; 10 children* helps to fasten in the minds of the children the data of the problem.

The Standardisation of Primary School Methods

The problem of the standardisation of Primary School methods is one which deserves the serious consideration of teachers. It is not for one moment suggested that Primary School methods should be standardised throughout the whole country, but educationists might well think upon the problem of whether or not such methods could be standardised within the area covered by a county borough or within an administrative district of one of the larger county authorities. By this means, methods would be standardised in Primary Schools which contribute to a group of Secondary Schools. The advantages of such standardisation appear to be two-fold; viz. (i) The methods recommended, arising out of teacher discussion and study, are certain to be the methods which have been proved in experience. (Note here that the methods decided on must be recommended by the teachers and must not be dictated to them.) (ii) The discontinuity of method between school and school within an area would be eliminated. This, as Schonell[1] has shown, is a cause of backwardness, for under present circumstances children moving from school to school within an area may miss vital steps while, as a result of transfer from Primary to Secondary School, a child is sometimes puzzled by being shown a second method before he has thoroughly mastered the first.

It is admitted, however, at once that there are also two disadvantages. First there is a slight loss of individual freedom for teachers. While not wishing to detract in any way from the freedom which the teacher enjoys and which must be maintained, it is thought that in this case the benefit brought to the pupils would outweigh the slight disadvantage suffered by the teacher. Second, in the case of a particular child, there may be an emphasis on an unsuitable method. Even here on balance the standardisation of method in the main brings benefits, because under present conditions the many and varied methods which the teachers use are unsuited to some children.

Revision

It is necessary to stress the importance of thorough revision of the number work that has been taught. Experimental psychology has shown that when such material as nonsense syllables, lists of words, poetry, and so on has been learnt and the learner tested for retention some hours or days later, there is a very rapid loss at first and a more gradual decline later. Material with meaning, e.g. lists of words, poetry, and so on, is forgotten more slowly than nonsense material. From the results of such experiments it is concluded that the first revision of a piece of new number work should come soon after learning. The ideal would probably be to revise a new rule on the following day, then after an interval of two days, and again after an interval of a week, and so on. In practice the ideal cannot be realised, but the important thing for the teacher to remember is that there must be revision of a new rule soon after it is first introduced, and then further revisions at increasing intervals of time. Both in oral and in written work examples of earlier work should frequently be included.

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[1] Schonell, F. J.: *Diagnosis of Individual Difficulties in Arithmetic*, p. 69 (Oliver & Boyd, 1949).

Assessment of Progress

A teacher must assess the progress that is being made by his pupils. This can be done in the following ways:

1. Frequent oral and written exercises to be worked by the pupils. Written work, especially, makes it clear to both teacher and pupil what parts of the work have been imperfectly learnt or forgotten. The correction must be thorough if the interest is to be maintained and the teacher to discern exactly the form of error. On the other hand, correction in the sense of putting right what is wrong is essentially the pupil's business.

2. The use of diagnostic[1] and standardised attainment tests, such as those of Schonell. In the case of standardised attainment tests teachers should satisfy themselves as to their reliability, and also of the size of the sample population on which the test norms were obtained. With the aid of good standardised tests, a check on the progress of children can be made from time to time judged against external criteria. On the other hand, too much time should not be spent on using standardised tests; the use of two such tests a year is probably adequate. In this connection the teacher might keep in mind the story of the camel that received the consent of its master to put its nose into the tent for warmth, and later occupied the entire tent to the exclusion of its master.

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[1] The great value of the diagnostic type of test, which will indicate which of the basic facts and basic processes remain unknown, is stressed.

CHAPTER XVII

BACKWARDNESS IN ARITHMETIC

It is interesting to note that *backward* is a term used only in connection with basic subjects. We do not speak of a child as 'backward' in some of the other subjects of the curriculum, but use such words as untidy, clumsy, inartistic, and so on. Probably this is because backwardness in some of these other subjects is not so serious a matter for purposes of living as backwardness in the basic skills.

As has already been shown in Chapter XV, ability in arithmetic depends to a considerable extent on general intelligence, so that it necessarily follows that those children of low intelligence quotient are likely to fall behind in the subject. Innate causes of backwardness are, more or less, beyond the power of the teacher to remedy. It is possible that work which is closely allied to the child's interests and everyday experience may help to overcome such things as dislike of material, but, generally speaking, when backwardness is due to intellectual causes there is not much the school can do about it except to make the school work as stimulating as possible.

As Schonell[1] has shown, a certain amount of backwardness is found to be due not to deficiency in 'g' at all, but to such causes as poor home conditions, for example. Children who come from homes where there is no opportunity to join in informal number activities—e.g. counting, weighing, measuring, and so on—are always at a disadvantage when they come to the more formal work in arithmetic. Other children arrive at school tired, not having had enough sleep for one reason or another, and are quite unable to give that degree of attention necessary for success in number work. Those who are away from school intermittently or for prolonged periods nearly always fall behind, for arithmetic depends more than any other subject on the mastery of a series of progressive steps. A child who is kept at home often passes some of the time reading, but seldom doing suitable (graded) arithmetic. In these days, when enthusiasm for Grammar School places is so great, young children are frequently forced beyond their natural pace by over-ambitious parents; emotional attitudes are set up and the children develop a dislike for arithmetic.

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[1] Op. cit.

School conditions, no less than home conditions, are frequently responsible for retardation in arithmetic. One of the most frequent causes is discontinuity between school and school, department and department, or even between class and class. In some schools children are allowed to use apparatus or other 'crutches', while in others they are denied such helps. A child passing from one school to another in such circumstances may find himself completely confused. The same result follows from the use of different teaching methods.

Too rapid promotion nearly always affects work in arithmetic more than in any other subject. Children are sometimes promoted by reason of their success in English subjects (particularly reading in the Primary School) and, as a result, essential steps in the arithmetical processes are not taught them or are taught only inadequately; and backwardness results.

Duller pupils are sometimes more backward than they would otherwise be as a result of over-emphasis on processes. 'Reasons why' are not necessary with dull pupils. They should be taught the few essential processes they need to know mechanically. Many adults use arithmetical processes without understanding the principles on which they depend. How many adults, for example, could explain why—in the process of division by a fraction—the divisor is turned upside down and the process turned into multiplication?

Because of the objectivity of arithmetic, too extensive a syllabus is often attempted, or the examples are badly graded, or the work proceeds too quickly, e.g. commencing a new step before the previous one is thoroughly mastered, or teaching two similar but not identical processes at the same time. All these tend to cause backwardness in individual cases or to aggravate any backwardness that already exists.

Schonell[1] contends that emotional causes of backwardness are the most potent of all. Early failure, he says, produces a feeling of confusion and continued failure a feeling of complete inadequacy. Unless such emotional tensions are relieved, a child feels completely overwhelmed and altogether unable to manage simple arithmetic.

To deal at all adequately with backwardness, individual treatment is necessary. The child must be taught by a sympathetic teacher, and the teaching done in a calm, unhurried atmosphere. Diagnostic tests[2] are necessary in the first instance to reveal actual weaknesses, and these, when discovered, must be remedied by the use of much concrete material, 'crutches', and work with easy figures. To this end, work with card games and arithmetical jigsaw puzzles is useful. As much oral work as possible should be carried out in which the number facts are related to real-life situations, e.g. delivering milk bottles, arranging flowers in vases, winning conkers, giving away cigarette cards, and so on.

Improvement in problem work is very difficult to accomplish because problem work depends more than any other kind of arithmetic on 'g,' as has already been shown. Some progress does usually follow improvement in reading, elimination of errors in the basic number facts, and the careful grading of examples, two-step problems following one-step problems, and so on.

The following practical suggestions have been found to minimise backwardness in arithmetic:

1. Where there is a considerable spread of arithmetical attainment in ordinary classes, backward scholars receive most help from a system of sets, i.e. they 'slide' for arithmetic.

2. Arithmetical material and processes should be constantly applied to real-life situations.

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[1] Op. cit.

[2] E.g. Schonell's *Diagnostic Tests in Arithmetic* (Oliver & Boyd).

3. Work on money, length, area, weight, and capacity should always be preceded by practical work in which the units themselves are handled.

4. Work should be motivated by visual records, speed tests, the use of easy examples, much oral work, and a sympathetic attitude on the part of the teacher.

5. All useless processes and unnecessarily large numbers should be eliminated.

6. We must accept the fact that the performance of dull children is often very limited. The school authorities must be content with a far less extensive syllabus for such children.

No attempt has been made in this chapter to give actual graded examples suitable for use with backward children, as so much depends upon individual abilities, needs, and background.

The above discussion of backwardness in arithmetic has generally followed the lines suggested by Schonell. From recent work, however, it seems that the causes of backwardness are more complex than he indicated. It is suggested, therefore, that the majority of children who are very backward in number work at about nine years of age should be looked upon as slow learners regardless of the intelligence quotient. Only a few will make rapid progress under special teaching. Accordingly readers should consult books that deal specifically with the teaching of slow-learning children. For example, see Tansley, A. E., and Gulliford, R.: *The Education of Slow Learning Children* (Routledge and Kegan Paul, 1960). Also, Cleugh, M. F. (editor): *Teaching the Slow Learner in the Primary School* (Methuen, 1961).

CHAPTER XVIII

THE ORGANISATION OF CLASSES AND WORK

THE most economical method of teaching arithmetic is that in which the class is treated as a unit. In recent years, following the findings of educational psychology particularly in the field of individual differences, class teaching has been subjected to a good deal of criticism. Up to a point this is inevitable. Where the needs of individual children are completely overlooked and where any individual child is forced to conform to some preconceived pattern of performance, the teaching method leaves much to be desired. Any well-planned curriculum and any well-thought-out teaching method will always leave the way open for the development of a particularly bright child or for the more detailed oversight of a less able one. But we should not suppose that good class teaching makes such individual attention impossible. Provided the attitude of the teacher is sympathetic and encouraging, and provided, too, that the worked is planned in graded steps, the less able children often gain much by being taught alongside the brighter—'iron sharpeneth iron'.

In many schools at the present time arithmetic is taught on the basis of group work, particularly if the school is not streamed. That is to say, a class is divided—after suitable classifying tests[1]—into groups of varying arithmetical ability. Such groups are taught in rotation and provided with sufficient exercises to keep them occupied until such time as the teacher is able to proceed to the next step. Such grouping does of course enable a child to get more individual attention—but still, be it noted, not complete individual attention—and it does, to some extent, allow children to work at a rate which is more closely related to their capacities. It is not necessary, when such a system of grouping is used, for the groups to be taught in strict rotation; the recurrence of a 'teaching' period can be determined by the teacher on the spot after a consideration of the amount of progress made by the group in question. Nor is it necessary for any attempt to be made to bring all groups to the same point eventually. The speed with which the work is undertaken can be determined entirely by the ability of the pupils within a particular group. While the group system of teaching has many advantages, it should not be overlooked that the members of a group will almost inevitably have to work more exercises on a particular topic than they really need to do, especially when there is a slow group in the class. The system also suffers from the defect that the children in any one group are somewhat out of touch with their teacher for longer periods than may be desirable.

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[1] E.g. Schonell's diagnostic tests.

'Sliding' for arithmetic is a method of organisation favoured by some head teachers. Here, children throughout the school, or maybe part of the school if the sliding is not to be complete, are tested for arithmetical ability and attainment. On the results of this testing the pupils are divided up into 'sets'. The set in which a particular child is found may not bear very much relation to the class in which he works for subjects other than arithmetic. In most cases, however, a child will not slide very far. The obvious advantage of this system is that the classes or sets are more homogeneous arithmetically and can therefore be taught as a larger unit.

A method of organisation which, on the surface, appears to approach perfection, viz. individual assignments, is not, in our opinion, very suitable for use with small children. Pupils of Primary School age need their work arranged in a series of carefully graded steps; steps based on a thorough analysis of all processes. To draw attention to and to emphasise this fact is one of our main purposes in writing this book. This means, however, that the children will need help from their teacher at frequent intervals, and a teacher with a large class of small children cannot hope to give the attention necessary when every child is working as an individual. Enthusiasts for individual work should not overlook the fact that while the staffing ratio is less than one teacher to one child, an individual child is never able to work at precisely that rate which his ability would suggest he should. It is a consoling thought that what little is lost by not being able to work at this precise rate is more than made up for by the social benefits that follow from working with a group of other children of about the same age and ability.

It should always be borne in mind that, for the most part at any rate, children do not learn arithmetic because they want to learn it; they learn it because they are taught and they are taught because they cannot get far in a modern community without some knowledge of it. It is something which, simply, is necessary for living in the modern world. If that is so, we should get on with learning and the teaching of the subject by the most effective means possible; and that involves work. It is our belief that arithmetic in school should at all times be related to real life; that the children should, as far as possible, be brought into contact with real things; that where it is desirable a practical approach should be made. But it is our belief, too, that a good deal of arithmetic is not learnt that way; it is learnt with pens and paper in the classroom and by applying oneself diligently to a task.

Class teaching in arithmetic still has an important part to play. Some homogeneity is of course desirable (a completely homogeneous class would surely be an insufferably dull unit to teach), and some individual attention is necessary.

A word about the marking of arithmetic may not be out of place. Every exercise should be marked—but not necessarily by the teacher. Ideally a child should do an exercise and then, straightway, get it marked. If correct, he should proceed to the next exercise; if not, it should be worked again. This ideal is, however, difficult to attain in practice. Two methods of marking have been found useful in our experience. In the first, several answer books are available in the classroom, and the pupils are told that when they have completed, say, three exercises they may consult the answer book. If the examples have been worked correctly, they proceed; otherwise the exercises are corrected. The teacher is available throughout the lesson for advice and help. It is of course necessary for the exercise-books to be collected from time to time to be looked through by the teacher.

In the second method the teacher starts the marking almost as soon as the pupils commence work. It is better that he should move among the pupils rather than that the pupils should line up at his desk; the latter arrangement wastes so much time. On his way round the class he is able to deal with individual difficulties, and often prevent mistakes being made. At the end of the period the answers are given so that the pupils may mark for themselves those exercises which the teacher has not seen. Such exercises will of course be seen on the next round.

In conclusion, it should be said that in the periods set aside for arithmetic there should be a proper balance between oral work and discussion involving concepts and principles, mental work, mechanical work, and problem work.

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